Section 7.4 - Confidence Intervals for the Variance and Standard Deviation of a Normal Population

Proposition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Then the estimator

$$\hat{\sigma}^2 = S^2 = \frac{\Sigma(X_i - \overline{X})^2}{n - 1}$$

is unbiased for estimating σ^2.

Definition

The **standard error** of an estimator $\hat{\theta}$ is its standard deviation $\sigma_{\hat{\theta}} = \sqrt{V(\hat{\theta})}$. It is the magnitude of a typical or representative deviation between an estimate and the value of θ.

Suppose that instead of estimating the population mean μ, we wish to estimate the population *variance* σ^2.

Let $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$ be the sample variance from a random sample of size n taken from a $N\left(\mu,\sigma^2\right)$ distribution. Then a $(1-\alpha)$ confidence interval for σ^2 is

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right]$$

where χ^2 has $n-1$ degrees of freedom.

Note: If an interval for *standard deviation* is desired, take the square root.

Example: A sample of $n=16$ completion times for a particular task for a lab technician led to a sample mean of 4.3 minutes and a sample standard deviation of 0.6 minutes. Determine a 95% confidence interval for the standard deviation of her completion time.

$\alpha = 1-.95 = .05 \quad \alpha/2 = .025$

$$\left[\frac{(n-1)S^2}{\chi^2_{\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right]$$

CI for variance:

$$\left[\frac{(16-1)(.6)^2}{\chi^2_{.025,15}}, \frac{(16-1)(.6)^2}{\chi^2_{.975,15}}\right]$$

$$\left[\frac{15(.6)^2}{27.488}, \frac{15(.6)^2}{6.262}\right] = [.1964, .862]$$

CI for st.dev: $\sigma \in [.443, .928]$

R-studio: (area to left)

qchisq(.975, 15) = 27.488

qchisq(.025, 15) = 6.262