Section 4.4 - The Exponential and Gamma Distributions

The Exponential Distribution

X is said to have an ***exponential distribution*** with parameter $\underline{\lambda}$ ($\lambda > 0$) if the pdf of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$P(X \leq a) = F(a) = \int_0^a \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^a = -e^{-a\lambda} - -e^{0}$$

Note: for the exponential distribution the <u>mean is 1/λ</u> .

$$E[X] = 1/\lambda$$

$$= 1 - e^{-a\lambda}$$

Ex: Suppose the time a child spends waiting at for the bus as a school bus stop is exponentially distributed with mean 3 minutes. Determine the probability that the child must wait at least 5 minutes on the bus on a given morning.

$$E[X] = 3 = 1/\lambda \rightarrow \lambda = 1/3$$

$$P(X \geq 5) = 1 - P(X \leq 5) = 1 - F(5)$$

$$= 1 - \int_0^5 \frac{1}{3} e^{-x/3}\, dx$$

$$= 1 - \left[-e^{-x/3}\right]_0^5$$

$$= 1 - \left[e^{-5/3} + 1\right] \approx .189$$

$$1 - F(5) = 1 - \left(1 - e^{-5/3}\right)$$

The Gamma Distribution

We say that X has a *Gamma Distribution* with parameters $\alpha > 0$ and $\beta > 0$ if X has p.d.f.

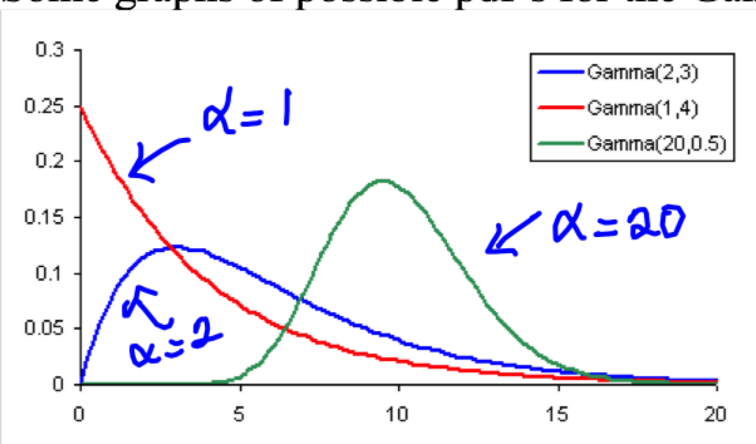
$$\boxed{f(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-y/\beta}, \ 0 \le y < \infty}$$

Note: Here $\boxed{\Gamma(\alpha) = \int_0^{\infty} w^{\alpha-1} e^{-w}\, dw}$ (The Gamma Function)

Interpretation of the Gamma Distribution:
If X has gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, then X represents the amount of time it takes to obtain α successes, where

$\beta = \frac{1}{\lambda}$, ($\lambda =$ expected number of occurrences is one time interval).

Some graphs of possible pdf's for the Gamma Distribution:



Special Cases of the Gamma Distribution:

1. If $\alpha = 1, \beta = \frac{1}{\lambda}$ then X is said to have *Exponential Distribution with parameter* λ.
2. If $\alpha = \frac{r}{2}, \beta = 2$ then X is said to have *Chi-square Distribution with parameter r*. In this case $f(y) = \frac{1}{\Gamma(r/2)2^{r/2}} y^{(r/2)-1} e^{-y/2}, \quad 0 \le y < \infty$.

Ex: Suppose X is a continuous rv with a gamma distribution with $\alpha = 3$. Find $P(4 \le X \le 6)$

$= P(X \le 6) - P(X \le 4)$

$.938 - .762 = \boxed{.176}$

(If discrete RV
$P(X < 4) = P(X \le 3)$)

Table A.4 The Incomplete Gamma Function

x \ α	1	2	3	4	5	6	7
1	.632	.264	.080	.019	.004	.001	.000
2	.865	.594	.323	.143	.053	.017	.005
3	.950	.801	.577	.353	.185	.084	.034
4	.982	.908	.762	.567	.371	.215	.111
5	.993	.960	.875	.735	.560	.384	.238
6	.998	.983	.938	.849	.715	.554	.394
7	.999	.993	.970	.918	.827	.699	.550
8	1.000	.997	.986	.958	.900	.809	.687
9		.999	.994	.979	.945	.884	.793
10		1.000	.997	.990	.971	.933	.870
11			.999	.995	.985	.962	.921
12			1.000	.998	.992	.980	.954
13				.999	.996	.989	.974
14				1.000	.998	.994	.986
15					.999	.997	.992