Continous Random Variables

Section 4.1 - Probability Density Functions

A random variable X is *continuous* if it may assume any value in an interval, and assumes any *particular* value with probability 0. $P(X=a)=0$

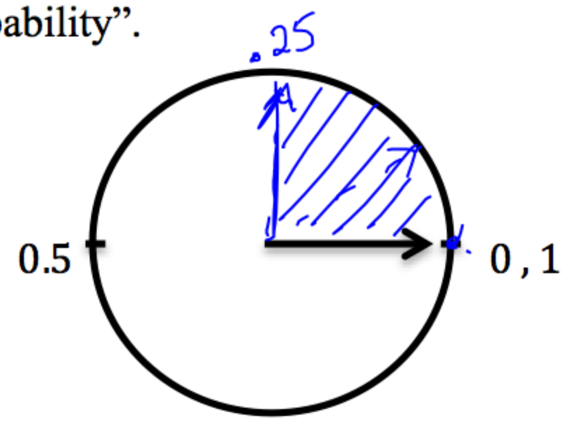
For a continuous random variable, the probability that X is in any given interval is the integral of the probability density function over the interval.

$f(x)$

$$P(a \le X \le b) = \int_a^b f(x)\,dx \leftarrow \text{area under } f(x)$$

Also, since f is a probability density function, it must be that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

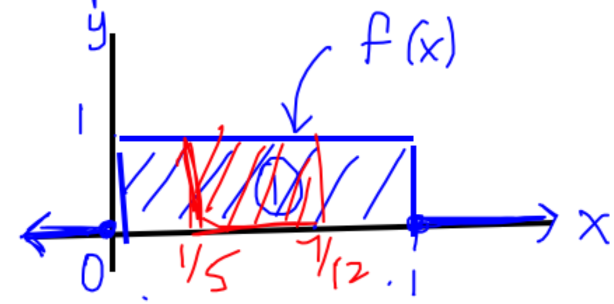
Example: Consider a spinner that, after a spin, will point to a number between zero and 1 with "uniform probability".



X = outcome of one spin

$P(0 \le X \le .25) = 1/4$

$P(0 \le X \le .5) = 1/2$



$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Define a probability density function for X, the result of the spin.

Determine $P\left(\frac{1}{5} \le X \le \frac{7}{12}\right)$. $= \int_{1/5}^{7/12} 1\,dx = x\Big|_{1/5}^{7/12} = 7/12 - 1/5 = \boxed{23/60}$

Note: For any continuous probability distribution, $P(X = x) = 0$ for all x.

Why is this?

$$P(X = x) = \int_x^x f(t)\,dt = 0$$

So this means that $\underbrace{P(a \le X \le b) = P(a < X < b)}$.

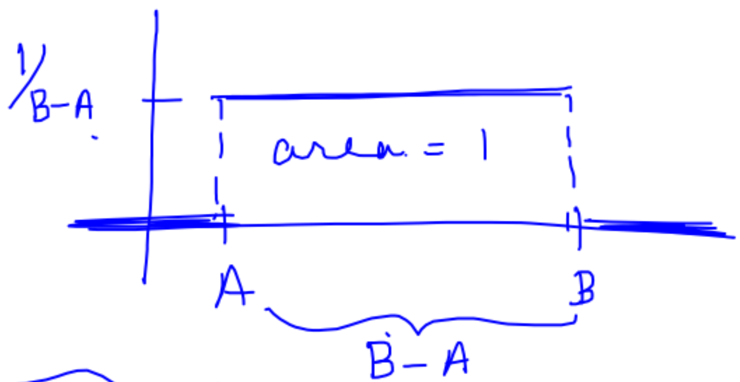
☆ for continuous RV only

The Uniform Distribution:

A continuous rv X is said to have ***uniform distribution*** on the interval $[A,B]$ if the pdf of X is given by

$$f(x;A,B)=\begin{cases}\dfrac{1}{B-A} & A\le x\le B\\ 0 & otherwise\end{cases}$$

What does the histogram of a uniform distribution look like?



Example: Suppose that the length X of the life (in years) of the field winding generator has a distribution that can be described by the pdf

$$f(x)=\frac{1.8x^{0.8}}{8^{1.8}}\exp\left[-\left(\frac{x}{8}\right)^{1.8}\right], \quad 0\le x<\infty$$

Determine the probability that a winding fails before the one year warranty expires on the machine

$$P(X<1)=P(X\le 1)=\int_0^1 f(x)\,dx=\int_0^1 \frac{1.8}{8^{1.8}}x^{.8}e^{-\left(\frac{x}{8}\right)^{1.8}}dx$$

$$\int_0^1 \underbrace{\frac{1.8}{8^{1.8}} x^{.8}}\, e^{-\left(\frac{x}{8}\right)^{1.8}} dx \longrightarrow \int_0^{-\left(\frac{1}{8}\right)^{1.8}} -e^u \, du$$

$$u = -\left(\frac{x}{8}\right)^{1.8}$$

$$du = -1.8 \left(\frac{x}{8}\right)^{.8} \cdot \frac{1}{8}\, dx$$

$$du = \frac{-1.8}{8^{1.8}} x^{.8}\, dx$$

$$x = 0 \Rightarrow u = -\left(\frac{0}{8}\right)^{1.8} = 0$$

$$x = 1 \Rightarrow u = -\left(\frac{1}{8}\right)^{1.8}$$

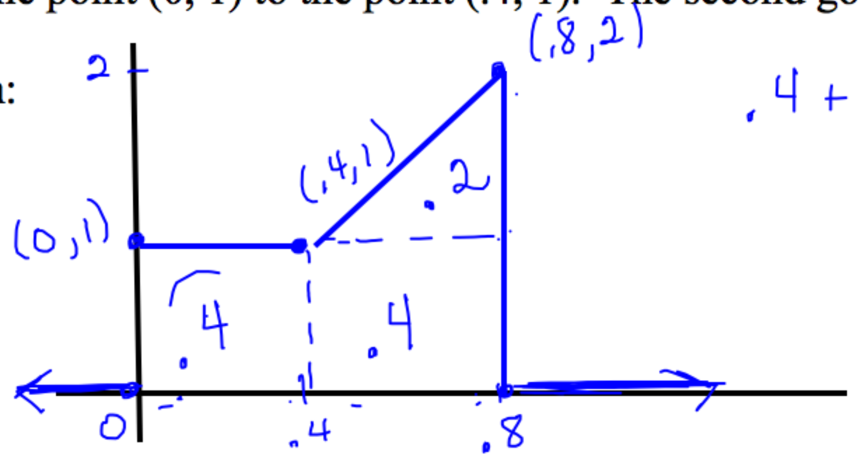
$$= -e^u \Big|_0^{-\left(1/8\right)^{1.8}}$$

$$= -\left[e^{-\left(\frac{1}{8}\right)^{1.8}} - e^0\right]$$

$$= \underline{\underline{.0234}}$$

Example: Think about a density curve for a continuous rv that consists of two line segments. The first goes from the point (0, 1) to the point (.4, 1). The second goes from (.4, 1) to (.8, 2) in the xy plane.

Sketch:



$.4 + .4 + .2 = 1$

What percent of observations fall below .4?

$P(X < .4) = .4$

What percent of observations lie between .4 and .8?

$P(.4 < X < .8) = .6$

What percent of observations are equal to .4?

$P(X = .4) = 0$

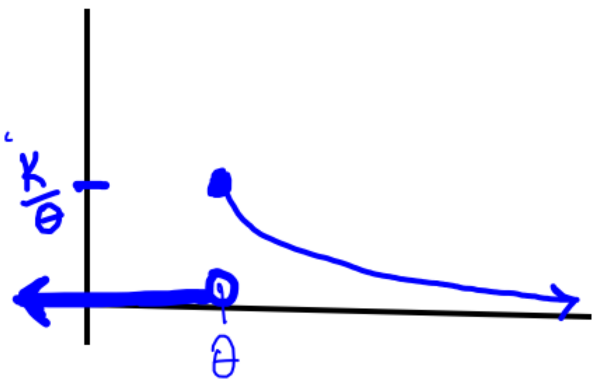
Example (#10 a and b from section 4.1)

10. A family of pdf's that has been used to approximate the distribution of income, city population size, and size of firms is the Pareto family. The family has two parameters, k and θ, both > 0, and the pdf is

$$f(x; k, \theta) = \begin{cases} \dfrac{k \cdot \theta^k}{x^{k+1}} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

a. Sketch the graph of $f(x; k, \theta)$.
b. Verify that the total area under the graph equals 1.
c. If the rv X has pdf $f(x; k, \theta)$, for any fixed $b > \theta$, obtain an expression for $P(X \leq b)$.
d. For $\theta < a < b$, obtain an expression for the probability $P(a \leq X \leq b)$.

a.



$$\frac{k \cdot \theta^k}{\theta^{k+1}} = \frac{k}{\theta}$$

b. $A = \int_{-\infty}^{\infty} f(x)\,dx = \int_{\theta}^{\infty} f(x)\,dx$

$$= \int_{\theta}^{\infty} \frac{k\theta^k}{x^{k+1}}\,dx = k\theta^k \int_{\theta}^{\infty} x^{-k-1}\,dx$$

$$= \lim_{b\to\infty} \left[k\theta^k \int_{\theta}^{b} x^{-k-1}\,dx \right] = \lim_{b\to\infty} \frac{\cancel{k}\theta^k x^{-k}}{-\cancel{k}} \Big|_{\theta}^{b}$$

$$= \lim_{b\to\infty} \left(-\underset{\to 0}{\theta^k b^{-k}} + \underline{\underline{\theta^k \theta^{-k}}} \right) = 0 + 1 = \boxed{1}$$

Section 4.2 - Cumulative Distribution Functions and Expected Values

Definition: If X is a random variable with probability density function f, then the *cumulative distribution function* (abbreviated c.d.f) is the function given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(w)\, dw \qquad \frac{d}{dx}\int_{-\infty}^{x} f(w)\,dw = f(x) = F'(x)$$

Example: The spinner from the previous example (uniform on [0,1]) is spun 3 times, and 3 independent observations X_1, X_2, X_3 are taken.
Let Y equal the maximum value of these three observations.
Determine the probability density function for Y.

$$Y = \max\{X_1, X_2, X_3\}$$

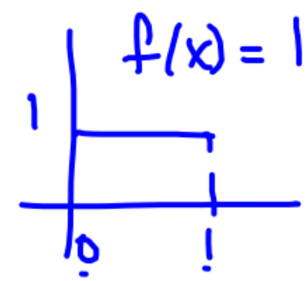
$$F(y) = P(Y \le y) = P(\max\{X_1, X_2, X_3\} \le y)$$

$$= P(X_1 \le y \text{ and } X_2 \le y \text{ and } X_3 \le y)$$

$$= P(X_1 \le y)\, P(X_2 \le y)\, P(X_3 \le y)$$

$$= \int_0^y f(x)\,dx \int_0^y f(x)\,dx \int_0^y f(x)\,dx$$

$$\left[\int_0^y f(x)\,dx\right]^3 = \left[\int_0^y 1\,dx\right]^3 = y^3$$



$$F(y) = y^3 \Rightarrow f(y) = 3y^2 \quad 0 \le y \le 1$$

$\xrightarrow{}$ $E[x]$ of discrete RV $= \sum x_i \, p(x_i)$

$\boxed{\text{Expected Value}}$ of a Continuous Random Variable:

If X is a continuous random variable with pdf f, then

$$E[X]=\int_{-\infty}^{\infty} x \cdot f(x)\,dx \quad \text{and} \quad E[u(X)]=\int_{-\infty}^{\infty} u(x) f(x)\,dx$$

Variance:

The ***variance*** of a continuous rv X with pdf f and mean value μ is

$$V(X)=\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

Shortcut: $\boxed{V(X)=E[X^2]-E[X]^2}$

Example: For the spinner from the previous example, with Y the maximum of 3 spins, determine $E[Y]$ and $\operatorname{var}(Y)$.

$f(y) = 3y^2 \quad 0 \le y \le 1$

$E[Y] = \int_0^1 y f(y)\,dy = \int_0^1 y \cdot 3y^2\,dy = \int_0^1 3y^3\,dy = \frac{3}{4} y^4 \Big|_0^1$

$= \boxed{3/4}$

$V[Y] = E[Y^2] - (E[Y])^2 = \int_0^1 y^2 \cdot 3y^2\,dy - (3/4)^2$

$= 3/5 - 9/16 = \boxed{.0375}$

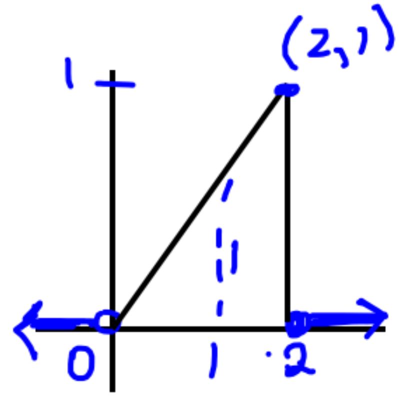
Example: (Exercise 11 from 4.2)

$f(x) = F'(x)$

11. Let X denote the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf is

$$P(X \le x) = F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{4} & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

$$f(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$



Use the cdf to obtain the following:

a. $P(X \le 1) = F(1) = 1/4$

b. $P(.5 \le X \le 1) = P(X \le 1) - P(X \le .5) = F(1) - F(.5) = 1/4 - 1/16 = 3/16$

c. $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - F(1.5) = 1 - .5625 = .4375$

d. The median checkout duration $\tilde{\mu}$ [solve $.5 = F(\tilde{\mu})$]

e. $F'(x)$ to obtain the density function $f(x)$

f. $E(X)$

g. $V(X)$ and σ_X

h. If the borrower is charged an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $E[h(X)]$.

$$E[X^2] = \int_0^2 x^2 f(x)\,dx = \int_0^2 x^3/2\,dx = \cdots = 2$$

d) $.5 = F(\tilde{\mu}) = \dfrac{X^2}{4}$

$2 = x^2$

$x = \sqrt{2}$

$\tilde{\mu} \approx 1.414$

f) $E[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x \cdot \dfrac{x}{2}\,dx = \dfrac{x^3}{6}\Big|_0^2 = 4/3 = 1.\bar{3}$

g) $V[X] = E[x^2] - (E[X])^2 = 2 - (4/3)^2 = .222$

$\sigma = .471$