15.2 – Evaluation by Repeated Integrals

If the region Ω is given by \( a \leq x \leq b, \ \phi_1(x) \leq y \leq \phi_2(x) \) (this is called a Type I region), then

\[
\int_{\Omega} f(x, y) \, dx \, dy = \int_{a}^{b} \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) \, dy \, dx
\]

If the region Ω is given by \( c \leq y \leq d, \ \psi_1(y) \leq x \leq \psi_2(y) \) (this is called a Type II region), then

\[
\int_{\Omega} f(x, y) \, dx \, dy = \int_{c}^{d} \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) \, dx \, dy
\]

Applications of double integrals include

Volume: \( V = \iiint_{\Omega} f(x, y) \, dx \, dy \), \( f(x, y) > 0 \), \( f(x, y) \) is top and \( \Omega \) is base

Area: \( A = \iint_{\Omega} dx \, dy \), \( \Omega \) is region to find area of

As well as Mass of a Plate and Center of Mass (later).
Examples:
1. Evaluate $\int\int_\Omega x^3 y \, dx \, dy$ taking $\Omega : 0 \leq x \leq 1, 0 \leq y \leq x$
2. Evaluate \[ \int_{\Omega} \cos(x+y) \, dx \, dy \] taking \( \Omega : 0 \leq x \leq \frac{\pi}{2}, \ 0 \leq y \leq \frac{\pi}{2} \)
3. Evaluate \( \int_{\Omega} (x^4 + y^2) \, dx \, dy \) taking \( \Omega \) is the region bounded between 
\( y = x^3 \) and \( y = x^2 \).
4. Calculate by double integration the area bounded by the curves \( y = x \) and \( x = 4y - y^2 \).
5. Give the formula for the volume under the paraboloid $z = x^2 + y^2$ within the cylinder $x^2 + y^2 \leq 1, \quad z \geq 0$ using double integrals.
6. Calculate \( \int_{0}^{2} \int_{0}^{4} 2x \cos(y^2) \, dy \, dx \) by changing the order of integration.
7. Find the volume of the solid bounded by the coordinate planes and the plane \( \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1 \)
15.3 – Polar Coordinates

Given \( F = F(r, \theta) \) continuous on 
\[ \Gamma: \quad a \leq r \leq b, \quad \alpha \leq \theta \leq \beta \]

\[
\iint_{\Gamma} F(r, \theta)r \, dr \, d\theta = \int_{\alpha}^{\beta} \int_{a}^{b} F(r, \theta) \, r \, dr \, d\theta
\]

And the double integral of \( F(r, \theta) \) over the polar region 
\[ \Omega: \quad \alpha \leq \theta \leq \beta, \quad \rho_1(\theta) \leq r \leq \rho_2(\theta) \]

\[
\iint_{\Omega} F(r, \theta)r \, dr \, d\theta = \int_{\alpha}^{\beta} \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) \, r \, dr \, d\theta
\]

<table>
<thead>
<tr>
<th>Rectangular</th>
<th>Polar</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A = \iint_{\Omega} dx , dy )</td>
<td>( A = \int_{\alpha}^{\beta} \int_{a}^{b} r , dr , d\theta )</td>
</tr>
<tr>
<td>( V = \iint_{\Omega} f(x, y) , dx , dy )</td>
<td>( V = \int_{\alpha}^{\beta} \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) , r , dr , d\theta )</td>
</tr>
</tbody>
</table>

Examples:

1. \( \iint_{0}^{2\pi} \int_{0}^{2} r^2 \, dr \, d\theta = \)
   a) Evaluate

b) Express the double integral in rectangular coordinates
2. Evaluate: \( \int_0^\pi \int_0^{2\sin \theta} r \, dr \, d\theta \)
Note: If we are integrating over a circle or part of a circle (integrand involves $x^2 + y^2$) we can use polar integrals to solve.

Examples:

3. Evaluate $\int_0^{\sqrt{2}} \int_y^{\sqrt{2}} (x^2 + y^2)^{3/2} \, dx \, dy$ by changing to polar coordinates.
4. Find the volume of the solid bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the $xy$-plane.
5. Find the volume of the solid bounded above by the paraboloid \( z = x^2 + y^2 \), below by the \( xy \)-plane and on the sides by the cylinder \( x^2 + y^2 = 2y \).
## 15.4 – Applications of Double Integration

<table>
<thead>
<tr>
<th></th>
<th>Rectangular</th>
<th>Polar</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Volume</strong></td>
<td>$\int!!!!\int_\Omega f(x,y),dx,dy$</td>
<td>$\int!!!!\int_\Gamma F(r,\theta)r,dr,d\theta$</td>
</tr>
<tr>
<td><strong>Area</strong></td>
<td>$\int!!!!\int_\Omega 1,dx,dy$</td>
<td>$\int!!!!\int_\Gamma r,dr,d\theta$</td>
</tr>
<tr>
<td><strong>Mass</strong></td>
<td>$M = \int!!!!\int_\Omega \lambda(x,y),dx,dy$, $\lambda = \text{density function}$</td>
<td>$\sim$</td>
</tr>
<tr>
<td><strong>Center of Mass</strong></td>
<td>$x_M = \frac{\int!!!!\int_\Omega \lambda(x,y)x,dx,dy}{M}$, $y_M = \frac{\int!!!!\int_\Omega \lambda(x,y)y,dx,dy}{M}$</td>
<td></td>
</tr>
</tbody>
</table>

Example:

Find the mass and center of mass where $\Omega$ is the triangle with vertices (0,0), (1, 3) and (1, 5). $\lambda(x, y) = xy$
15.5 – Triple Integrals

The biggest difference between $\int \int f(x, y) dxdy$ and $\int \int \int f(x, y, z) dxdydz$ is that instead of working with two variables continuous over a plane, we are working with three variables over a continuous three dimensional space.

Integration over a “Box”:

Given $f = f(x, y, z)$ is continuous on the rectangular “box” $B$, where

$B: \ a_1 \leq x \leq b_1, \ a_2 \leq y \leq b_2, \ a_3 \leq z \leq b_3$
15.6 – Triple Integrals

Integration over an arbitrary solid:

\[ \iiint_S f(x, y, z) \, dx \, dy \, dz \]

Applications:

1. “Volume” of “hypersolid” = \[ \iiint_S f(x, y, z) \, dx \, dy \, dz \]

2. Volume of \( S \) = \[ \iiint_S dx \, dy \, dz \]

Reduction to a repeated integral

1. Type I: \( a \leq x \leq b \quad \phi_1(x) \leq y \leq \phi_2(x) \quad \psi_1(x, y) \leq z \leq \psi_2(x, y) \)

2. Type II: \( c \leq y \leq d \quad \phi_1(y) \leq x \leq \phi_2(y) \quad \psi_1(x, y) \leq z \leq \psi_2(x, y) \)

3. Type III:

4. ......

Examples:

1. Evaluate: \[ \int \int \int_{0 \leq x \leq 1} \int_{0 \leq y \leq 1-x} \int_{0 \leq z \leq y} 4z \, dz \, dy \, dx \]
2. Calculate $\iiint_T z\,dx\,dy\,dz$ where $T$ is the tetrahedron in the first octant bounded by the plane $x + y + z = 1$.
3. Find the volume of the solid bounded above by the plane \( y + z = 2 \), below by the \( xy \)-plane, and on the sides by \( x = 6 \) and \( y = \sqrt{x} \)
4. Set up the integral to find the volume of the solid bounded above by the hemisphere 
\[ z = \sqrt{4 - x^2 - y^2} \] and below by the cone \[ z = \sqrt{3x^2 + 3y^2} \]