- LecAlt07 was late, but is now under Wednesday 10/11.
- Almost all Test 1 FR are in the "Grades" tab on CASA.
- I will use your Final Exam score to replace one lower test score.
- I will have Test 1 FR for student pickup during my office hours, 2-4 p.m. Tuesdays and Thursdays in 207 PGH.
1. Parameterize the boundary of the region: \( D = \{(x, y): 0 \leq x \leq 2, 0 \leq y \leq 2x\} \)

   - \( C_1 \): \( x(t) = 2t, \quad y(t) = 0, \quad 0 \leq t \leq 2 \)
   - \( C_2 \): \( x(t) = 2, \quad y(t) = 4t, \quad 0 \leq t \leq 2 \)
   - \( C_3 \): \( x(t) = 2t, \quad y(t) = 4, \quad 0 \leq t \leq 2 \)

2. Choose A
15.7 - Maxima and Minima with Side Conditions

In section 15.6 we found absolute extrema of a function on a region that contained its boundary (like we did in Calc 1 when we used a closed and bounded interval). Now we will look at a different process – we will be optimizing a function $f$ subject to a constraint $g = 0$.

Method of Lagrange:

($\lambda$ is called the *Lagrange multiplier*)

Set your constraint function $= 0$ (that’s $g$)

Then solve the system:

$$\begin{align*}
\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\
g(x, y, z) &= 0
\end{align*}$$

Last, plug in all solutions $(x, y, z)$ into $f$ to identify max and min values (if they exist).
Level curves:
\[ f(x, y) = c \]

\[ C: g(x, y) = 0 \]

\[ \nabla g \parallel \nabla f \]

Figure 15.7.3
Examples:
1. Maximize $xy$ on the ellipse $4x^2 + 9y^2 = 36$.

Let $f(x,y) = xy$, $g(x,y) = 4x^2 + 9y^2 - 36$

$\nabla f(x,y) = (y, x)$, $\nabla g(x,y) = (8x, 18y)$

Set $\nabla f = \lambda \nabla g$:

$y = \lambda \cdot 8x$

$x = \lambda \cdot 18y$

$\Rightarrow \lambda = \frac{y}{8x}$ and $\lambda = \frac{x}{18y}$

Suppose $x = 0$. $\lambda = \frac{y}{8x}$ is invalid but $g(0, y) = 0$

becomes $4 \cdot 0^2 + 9y^2 = 36$ $\Rightarrow y^2 = 4$ $\Rightarrow y = \pm 2$

Can $f(0, 2) = 0 \cdot 2 = 0$ or $f(0, -2) = 0 \cdot (-2) = 0$ be the maximum value of $f(x,y)$ on this ellipse?

No, since there are obviously points in Quadrants I and III on this ellipse ($y = 0$).

We don't need to worry about $x = 0$ or $y = 0$ in THIS problem because of the above.
Last time, we saw that setting \( \frac{y}{8x} = \frac{x}{18y} \) gave \( 9y^2 = 4x^2 \) and 4 candidate points:

\[
\left( \frac{3}{\sqrt{12}}, \sqrt{2} \right), \left( -\frac{3}{\sqrt{12}}, \sqrt{2} \right), \left( -\frac{3}{\sqrt{12}}, -\sqrt{2} \right), \left( \frac{3}{\sqrt{12}}, \sqrt{2} \right)
\]

Max is \( f\left( \frac{3}{\sqrt{12}}, \sqrt{2} \right) = f\left( -\frac{3}{\sqrt{12}}, -\sqrt{2} \right) = 3 \)
2. Minimize \( x + 2y + 4z \) on the sphere \( x^2 + y^2 + z^2 = 7 \).

Let \( f(x,y,z) = x + 2y + 4z \), \( g(x,y,z) = x^2 + y^2 + z^2 = 7 \), \( g = 0 \) is

\[
\nabla f(x,y,z) = (1,2,4), \quad \nabla g(x,y,z) = (2x, 2y, 2z)
\]

\[
\nabla f = \lambda \nabla g \quad \Rightarrow \quad \begin{cases} 
1 = \lambda \cdot 2x \\
2 = \lambda \cdot 2y \\
4 = \lambda \cdot 2z
\end{cases}
\]

None of \( x, y, z \neq 0 \)

1: \( \frac{1}{2x} = \frac{1}{y} \Rightarrow y = 2x \)

2: \( \frac{1}{2y} = \frac{2}{z} \Rightarrow z = 2y \)

3: \( g = 0 \) \quad \Rightarrow \quad 7 = x^2 + y^2 + z^2 = x^2 + 4x^2 + 16x^2 = 21x^2
\]

\[
\Rightarrow \quad x^2 = \frac{7}{21} = \frac{1}{3} \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{3}}
\]

\[
y = 2 \cdot x = \pm \frac{2}{\sqrt{3}}, \quad z = 4x = \pm \frac{4}{\sqrt{3}}
\]

How many points? 2, if \( y = 2x \) and \( z = 4x \) then all of \( x, y, z \) must have the SAME sign.
2 points: \( \left( \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{4}{\sqrt{3}} \right) \) and \( \left( \frac{-1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-4}{\sqrt{3}} \right) \)

\[ f(x, y, z) = x + 2x - 4x \]

\[ \text{Min is } f\left( \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-4}{\sqrt{3}} \right) = \frac{1}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{16}{\sqrt{3}} = -\frac{21}{\sqrt{3}} = -7\sqrt{3} \]

\[ \text{Max is } +7\sqrt{3} \]
3. Find the points on the sphere \( x^2 + y^2 + z^2 = 1 \) that are closest to and farthest from the point (3, 1, 3).

Distance from \((x, y, z)\) to \((3, 1, 3)\) is \( \text{dist}(x, y, z) = \sqrt{(x-3)^2 + (y-1)^2 + (z-3)^2} \)

Let \( f(x, y, z) = (x-3)^2 + (y-1)^2 + (z-3)^2 \), \( g(x, y, z) = x^2 + y^2 + z^2 - 1 \)

\( \nabla f(x, y, z) = (2(x-3), 2(y-1), 2(z-3)) \), \( \nabla g(x, y, z) = (2x, 2y, 2z) \)

\( \nabla f = \lambda \cdot \nabla g \) : \( x-3 = \lambda \cdot x \) or \( x = \frac{3}{1-\lambda} \)
\( y-1 = \lambda \cdot y \) or \( y = \frac{1}{1-\lambda} \)
\( z-3 = \lambda \cdot z \) or \( z = \frac{3}{1-\lambda} \)

Similarly \( g = \frac{1}{1-\lambda} \), \( z = \frac{3}{1-\lambda} \) (or \( x = z \) and \( x = 3y \))

\( g = 0 \) : \( 1 = x^2 + y^2 + z^2 = \left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{3}{1-\lambda}\right)^2 \)

\( 1 = \frac{1}{(1-\lambda)^2} \left[ 3^2 + 1^2 + 3^2 \right] = \frac{19}{(1-\lambda)^2} \)

\( \Rightarrow (1-\lambda)^2 = 19 \Rightarrow 1-\lambda = \pm \sqrt{19} \)

So, \( x = \pm \frac{3}{\sqrt{19}} \), \( y = \pm \frac{1}{\sqrt{19}} \), \( z = \pm \frac{3}{\sqrt{19}} \)

How many points? 2, since \( x, y, \) and \( z \) get sign of \( 1-\lambda \)
Fixed point $(3, 1, 3)$

2 points: \( \left( \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}, \frac{3}{\sqrt{19}} \right) \) and \( \left( -\frac{3}{\sqrt{19}}, -\frac{1}{\sqrt{19}}, -\frac{3}{\sqrt{19}} \right) \)

- Closest
- Farthest

2D simplified view
15.8 – Differentials

Recall from Calc I:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

If \( h \) is small, \( f'(x) \approx \frac{f(x+h) - f(x)}{h} \)

\[ \Rightarrow \quad f'(x) \cdot h \approx f(x+h) - f(x) \]

\( df \), the \( \Delta f \), the
Differential of \( f \) Increment of \( f \)

When approximating:

\[ f(x_0 + h) \approx f(x_0) + f'(x_0) \cdot h \]

Now, for \( f(x, y) \), the differential \( df = \nabla f(x) \cdot \mathbf{h} \) or

\[ df = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \]

And for \( f(x, y, z) \)

\[ df = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial z} \Delta z \]

\[ f(x + \Delta x, y + \Delta y, z + \Delta z) \approx f(x, y, z) + \nabla f(x, y, z) \cdot (\Delta x, \Delta y, \Delta z) \]
Examples:
1. Find the differential $df$
   a. $f(x, y, z) = xy + yz + xz$. 
      \[ \nabla f(x, y, z) = (y + z, x + z, x + y) \]
      \[ \vec{h} = (\Delta x, \Delta y, \Delta z) \]
      \[ df = \nabla f(x, y, z) \cdot \vec{h} \]
      \[ df = (y + z) \Delta x + (x + z) \Delta y + (x + y) \Delta z \]
b. \( f(x, y) = \sin(x + y) + \sin(x - y). \)

\[ \nabla f(x, y) = (\cos(x+y) + \cos(x-y), \cos(x+y) - \cos(x-y)) \]

\[ df = (\cos(x+y) + \cos(x-y)) \Delta x + (\cos(x+y) - \cos(x-y)) \Delta y \]

Example of \((x, y)\) and \(\vec{h}\):

We want to find \(f(3.1, 1.9)\) and we know \(f(3, 2)\), \(\vec{h} = (3.1, 1.9) - (3, 2) = (0.1, -0.1)\)
2. Use differentials to approximate $\sqrt[4]{125} \cdot \sqrt[4]{17}$

Let $f(x,y) = \sqrt[4]{x} \cdot \sqrt[4]{y} = x^{\frac{1}{4}} \cdot y^{\frac{1}{4}}$

Want $f(125,17)$, we know $f(121,16) = 11 \cdot 2 = 22$

$121 + \Delta x = 125 \Rightarrow \Delta x = 125 - 121 = 4$

$16 + \Delta y = 17 \Rightarrow \Delta y = 17 - 16 = 1$

$\nabla f(x,y) = \left( \frac{1}{4}x^{-\frac{3}{4}}y^{\frac{1}{4}}, \frac{1}{4}x^{\frac{1}{4}}y^{-\frac{3}{4}} \right) = \left( \frac{\frac{1}{4}y^{\frac{1}{4}}}{2x^{\frac{1}{4}}}, \frac{x^{\frac{1}{4}}}{4y^{-\frac{3}{4}}} \right)$

$\nabla f(121,16) = \left( \frac{2}{2 \cdot 11}, \frac{11}{4 \cdot 8} \right) = \left( \frac{1}{11}, \frac{11}{32} \right)$

Now, $\sqrt[4]{125} \cdot \sqrt[4]{17} = f(125,17) \approx f(121,16) + \nabla f(121,16) \cdot (4,1)$

$= 22 + \left( \frac{1}{11}, \frac{11}{32} \right) \cdot (4,1)$

$= 22 + \frac{4}{11} + \frac{11}{32} \cdot 4$

$= 22 + \frac{24}{32} \cdot \frac{11}{32} \approx 22.707$

Calculator: $\sqrt[4]{125} \cdot \sqrt[4]{17} \approx 22.702$
15.9 - Reconstructing a Function from its Gradient

The gradient of a function \( f(x, y) \) is \( \nabla f(x, y) = \frac{\partial f}{\partial x}(x, y) \hat{i} + \frac{\partial f}{\partial y}(x, y) \hat{j} \)

So if we have a (gradient) vector in the form

\[
P(x, y)i + Q(x, y)j
\]

We need \( P(x, y) = f_x(x, y) \)

and \( Q(x, y) = f_y(x, y) \)

We can begin reconstruct the function by setting \( P = \partial f / \partial x \) and \( Q = \partial f / \partial y \)

Now when integrating \( \partial f / \partial x \) with respect to \( x \), remember that \( y \) is a constant!
Procedure:

1. Integrate $P(x,y)$ with respect to $x$. ($\int P(x,y) \, dx$)
   Keep in mind the integration "constant" could be a function of $y$. Call this $\phi(y)$ instead of $C$.

2. Take the derivative of result of 1 w.r.t. $y$ and compare to $Q(x,y)$.

3. Find $\phi(y)$, use this to find $f(x,y)$. 
Example: \( xy^2 \mathbf{i} + x^2y \mathbf{j} = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j} \)

\[
\int P \, dx = \int x y^2 \, dx = \frac{1}{2} x^2 y^2 + \phi(y)
\]

Let \( f(x,y) = \frac{1}{2} x^2 y^2 + \phi(y) \)

\[
f_y(x,y) = \frac{1}{2} x^2 \cdot 2y + \phi'(y) = x^2 y + \phi'(y)
\]

Set \( f_y = Q \Rightarrow x^2 y + \phi'(y) = x^2 y \)

\[\Rightarrow \phi'(y) = 0\]

\[\Rightarrow \phi(y) = C \quad \text{\(A\ true\ constant!\)}
\]

Thus, \( f(x,y) = \frac{1}{2} x^2 y^2 + C \)