- Test 1 FR grades should be posted by late Friday.

- I will use the Final Exam grade to replace one lower test grade.
THEOREM 15.3.4  CHAIN RULE (ALONG A CURVE)

If $f$ is continuously differentiable on an open set $U$ and $\mathbf{r} = \mathbf{r}(t)$ is a differentiable curve that lies in $U$, then the composition $f \circ \mathbf{r}$ is differentiable and

$$
\frac{d}{dt}[f(\mathbf{r}(t))] = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).
$$

\[
\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\
= \left( \frac{\partial f}{\partial x}(x(t), y(t)), \frac{\partial f}{\partial y}(x(t), y(t)) \right) \cdot \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \\
= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}
\]
And if

\[ u = u(x, y, z) \quad \text{where} \quad x = x(s, t) \quad y = y(s, t), \quad z = z(s, t) \]

Then

\[ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}, \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}. \]

\[ \mathbf{u}_s = u_x \cdot x_s + u_y \cdot y_s + u_z \cdot z_s \]
Example:

5. \( u = x^2 - xy + z^2 \) \( x = s \cos t \) \( y = \sin(t - s) \) \( z = t \sin s \). Find \( \frac{\partial u}{\partial t} \)

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 2x - y = 2s \cos t - \sin(t - s) \\
\frac{\partial u}{\partial y} &= -x = -s \cos t \\
\frac{\partial u}{\partial z} &= 2z = 2t \sin s
\end{align*}
\]

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} x_t + \frac{\partial u}{\partial y} y_t + \frac{\partial u}{\partial z} z_t
\]

\[
= (2s \cos t - \sin(t - s))(-s \sin t) + (-s \cos t) \cos(t - s)
\]

\[
+ 2t \sin s \cdot \sin s
\]

\[
= -2s^2 \cos t \sin t + s^2 \sin^2(t - s) - s \cos t \cos(t - s)
\]

\[
+ 2t \sin t \sin s
\]
1. Suppose that $f(x, y)$ is a differentiable function such that $\nabla f(0,1) = 2\mathbf{i} - \mathbf{j}$ and $\nabla f(1,0) = \mathbf{i} + 3\mathbf{j}$. Let $\mathbf{r}(t) = (e^t - 1)\mathbf{i} + (e^t - t)\mathbf{j}$ and define $g(t) = f(\mathbf{r}(t))$. Find $g'(0)$.

   a. 1
   b. 2
   c. 3
   d. -1
   e. None of these

2. Choose B
Suppose that $f(x, y)$ is a non-constant function that is continuously differentiable. That means $f$ is differentiable and its gradient $\nabla f$ is continuous. We saw last week that at each point in the domain $\nabla f$ (if $\neq 0$) points in the direction of the most rapid increase of $f$. Also, at each point of the domain, the gradient vector $\nabla f$ (if $\neq 0$) is perpendicular to the level curve of $f$ that passes through that point.

Why?

Let $\mathbf{r}(t)$ parameterize the level curve $f(x, y) = c$.

Thus, $\frac{dx}{dt} \left[\mathbf{r}(t)\right] = \frac{dx}{dt} \left[C\right]$.

$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$

For each $t$,

$\nabla f(\mathbf{r}(t)) \perp \mathbf{r}'(t)$

Therefore, at any point $(x_0, y_0)$ on the level curve,

$\nabla f(x_0, y_0)$ is normal to the curve.
Fact about 2-D vectors:

Given a vector $\mathbf{v} = (a, b)$, what can be said about $(b, -a)$?

$(b, -a) \perp (a, b)$ because

$(b, -a) \cdot (a, b) = b \cdot a - a \cdot b = 0$
The vector

\[ \mathbf{t}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) \mathbf{i} - \frac{\partial f}{\partial x}(x_0, y_0) \mathbf{j} \]

Is perpendicular to the gradient so it is one way to describe the **tangent vector** to the curve \( C \).

This representation is useful when we do not have or do not want to find a parameterization.

Why is this the tangent vector to a level curve containing \((x_0, y_0)\)?

Since \( \nabla f(x_0, y_0) \perp \) the curve

and \( \nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \mathbf{i} + \frac{\partial f}{\partial y}(x_0, y_0) \mathbf{j} \)

and \((b, -a)\) is always \( \perp \) to \((a, b)\)
The equation of the **tangent line** is:

\[
\left( \frac{\partial f}{\partial x} \right) (x_0, y_0) (x - x_0) + \left( \frac{\partial f}{\partial y} \right) (x_0, y_0) (y - y_0) = 0
\]

To \( f(x,y) = c \) at \((x_0,y_0)\)

1. Movement along line \( l \) from \((x_0, y_0)\) to \((x, y)\) is \( \mathbf{d} = (x-x_0, y-y_0) \)
   
   \( \nabla f(x_0, y_0) \parallel \mathbf{d} \Rightarrow \nabla f(x_0, y_0) \cdot \mathbf{d} = 0 \)
   
   \( \Rightarrow \left( \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right) \cdot (x-x_0, y-y_0) = 0 \)
   
   \( \Rightarrow \frac{\partial f}{\partial x}(x_0,y_0) (x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0) (y-y_0) = 0 \)

2. Use symmetric form of line through \((x_0, y_0)\) with \( \mathbf{d} = \nabla f(x_0, y_0) \)

   \[
   \frac{x-x_0}{\frac{\partial f}{\partial y}(x_0,y_0)} = \frac{y-y_0}{\frac{\partial f}{\partial x}(x_0,y_0)}
   \]
And the equation of the **normal line** is:

\[
\frac{\partial f}{\partial y} (x_0, y_0)(x - x_0) - \frac{\partial f}{\partial x} (x_0, y_0)(y - y_0) = 0
\]

Movement along line looks like

\[ \mathbf{d} = (x - x_0, y - y_0) \]

\[ \mathbf{N}(x_0, y_0) \perp \mathbf{d} \]

\[ \mathbf{N}(x_0, y_0) \cdot \mathbf{d} = 0 \]

\[
\left( \frac{\partial f}{\partial y}(x_0, y_0), -\frac{\partial f}{\partial x}(x_0, y_0) \right) \cdot (x - x_0, y - y_0) = 0
\]

\[
\frac{\partial f}{\partial y}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial x}(x_0, y_0)(y - y_0) = 0
\]
Example:
Write an equation for the tangent line and an equation for the normal line at point \( P \).

\[ x^2 + x\,y + y^2 = 7 \]

\[ P (1, -3) \]

We need a function so that this is a level curve.

Define \( f(x, y) = x^2 + x\,y + y^2 \), so our curve is \( f(x, y) = 7 \).

Notice that \( f(1, -3) = 1 + 1 \cdot (-3) + (-3)^2 = 1 - 3 + 9 = 7 \)

\( \nabla f(x, y) = (2x + y, x + 2y) \), \( \nabla f(1, -3) = (2 - 3, 1 - 6) = (-1, -5) \)

So, \((-1, -5)\) is normal to level curve @ \((1, -3)\)

\begin{align*}
\text{Tangent line} & : \\
-1(x - 1) - 5(y + 3) = 0 & \quad \Rightarrow \quad \vec{e}(1, -3) = (-5, 1) \\
-(x - 1) - 5(y + 3) = 0 \\
\text{Normal line} & : \\
-5(x - 1) + 1(y + 3) = 0 & \quad \Rightarrow \quad -5(x - 1) + (y + 3) = 0
\end{align*}
For \( f(x,y,z) \), the equation for the **tangent plane** is

\[
f_x(x_0, y_0, z_0)(x-x_0) + f_y(x_0, y_0, z_0)(y-y_0) + f_z(x_0, y_0, z_0)(z-z_0) = 0
\]

And the **normal line** is

\[
x = x_0 + \frac{\partial f}{\partial x}(x_0, y_0, z_0) t,
\]

\[
y = y_0 + \frac{\partial f}{\partial y}(x_0, y_0, z_0) t,
\]

\[
z = z_0 + \frac{\partial f}{\partial z}(x_0, y_0, z_0) t.
\]

\[\vec{N} = \nabla f(x_0, y_0, z_0)\]
Example:
Find an equation for the tangent plane and scalar parametric equations for the normal line at the point $P$.

$$x^3 + y^3 = 3xyz, \quad P(1, 2, \frac{3}{2})$$

Set $f(x, y, z) = x^3 + y^3 - 3xyz$, our surface is $f(x, y, z) = 0$

$$\nabla f(x, y, z) = \langle 3x^2 - 3yz, 3y^2 - 3xz, -3xy \rangle$$

$$\nabla f(1, 2, \frac{3}{2}) = \langle 3 - 9, 12 - \frac{9}{2}, -6 \rangle = \langle -6, \frac{15}{2}, -6 \rangle = -\frac{3}{2} \langle 4, -5, 4 \rangle$$

use $\vec{N} = (4, -5, 4)$

**Plane:** $4(x - 1) - 5(y - 2) + 4(z - \frac{3}{2}) = 0$

**Line:** $x = 1 + 4t, \quad y = 2 - 5t, \quad z = \frac{3}{2} + 4t$
Find the point(s) on the surface at which the tangent plane is horizontal.

\[ z - 2x^2 - 2xy + y^2 + 5x - 3y + 2 = 0 \]

What does it mean for a plane to be horizontal?
We will have a normal vector \( \mathbf{N} = (0, 0, c) = c \mathbf{k}, \ c \neq 0 \)

Let \( g(x, y, z) = z - 2x^2 - 2xy + y^2 + 5x - 3y + 2 \)
our surface is \( g(x, y, z) = 0 \)

\[ \nabla g(x, y, z) = (-4x - 2y + 5, -2x + 2y - 3, 1) \]

We need \((x, y, z)\) so that \( \nabla g(x, y, z) = (0, 0, 0) \)

Thus,

\[ \begin{align*}
-4x - 2y + 5 &= 0 \\
-2x + 2y - 3 &= 0 \\
-6x + 2z &= 0 \\
\end{align*} \]

\(-6x + 2z = 0 \Rightarrow -3x = -z \Rightarrow x = \frac{1}{3} \Rightarrow y = \frac{11}{6} \)

What is \( z \)? Set \( g(\frac{1}{3}, \frac{11}{6}, z) = 0 \) \( \Rightarrow z = -\frac{1}{12} \)

Our point is \( (\frac{1}{3}, \frac{11}{6}, -\frac{1}{12}) \)

The plane is \( z = -\frac{1}{12} \)
Here the surface is the graph of \( f(x,y) \), that is \( (x,y,f(x,y)) \)

Find the point(s) on the surface at which the tangent plane is horizontal.

\[
f(x,y) = 2x^2 + 2xy - y^2 - 5x + 3y - 2
\]

Where is \( z \) here? \( z = f(x,y) \)

For a plane, we need a 3-D normal vector, but \( \nabla f(x,y) \) is 2-D.

Let \( g(x,y,z) = z - f(x,y) \) (Our surface is \( g(x,y,z) = 0 \))

\[
g(x,y,z) = z - 2x^2 - 2xy + y^2 + 5x - 3y + 2
\]

\[
\nabla g(x,y,z) = (-4x - 2y + 5, -2x + 2y - 3, 1)
\]

\[
\nabla g(x,y,z) = (\frac{\partial g}{\partial x}(x,y), \frac{\partial g}{\partial y}(x,y), 1)
\]

**Key observation**: For a horizontal plane to graph of \( f(x,y) \) we need \( \nabla g(x,y,z) = (0,0,c) \Rightarrow \nabla f(x,y) = \vec{0} \)

This gives

\[
\begin{align*}
4x + 2y - 5 &= 0 \\
2x - 2y + 3 &= 0
\end{align*}
\]

Solve this to get \( x = \frac{1}{3}, y = \frac{11}{6}, f(\frac{1}{3}, \frac{11}{6}) = \frac{-1}{12} \)
Our point is \( \left( \frac{1}{3}, \frac{11}{6}, \frac{-1}{12} \right) \)

Our input point is \( \left( \frac{1}{3}, \frac{11}{6} \right) \)
3. Give the point where the tangent plane is horizontal for

\[ f(x, y) = 2x^2 + y^2 - 4x - 2y + 2 \]

\[ \nabla f(x, y) = (4x - 4, 2y - 2) = (0, 0) \]

\[ \Rightarrow 4x - 4 = 0 \quad \text{and} \quad 2y - 2 = 0 \]

\[ x = 1, \quad y = 1 \]

c) (1, 1)
14.5 - Local Extreme Values

**DEFINITION 15.5.1 LOCAL MAXIMUM AND LOCAL MINIMUM**

Let $f$ be a function of several variables and let $x_0$ be an interior point of the domain:

$f$ is said to take on a *local maximum* at $x_0$ if

$$f(x_0) \geq f(x) \quad \text{for all } x \text{ in some neighborhood of } x_0;$$

$f$ is said to take on a *local minimum* at $x_0$ if

$$f(x_0) \leq f(x) \quad \text{for all } x \text{ in some neighborhood of } x_0.$$

**THEOREM 15.5.2**

If $f$ takes on a local extreme value at $x_0$, then

either $\nabla f(x_0) = 0$ or $\nabla f(x_0)$ does not exist.
Critical points at which the gradient is zero are called **stationary points**. The stationary points that do not give rise to local extreme values are called **saddle points**.
Recall the 2nd Derivative Test:

We have a function \( f(x) \), \( c \) is a critical number.

If \( f'(c) = 0 \) and \( f''(c) > 0 \)
\[ f(c) \text{ is a local min} \]

If \( f'(c) = 0 \) and \( f''(c) < 0 \)
\[ f(c) \text{ is a local max} \]

If \( f'(c) = 0 \) and \( f''(c) = 0 \)
\[ f(c) \text{ could be a min, a max, or neither} \]
The test fails here!

Examples: \( f(x) = x^4 \), \( g(x) = -x^4 \), \( h(x) = x^3 \)
**THEOREM 15.5.3  THE SECOND-PARTIALS TEST**

Suppose that $f$ has continuous second-order partial derivatives in a neighborhood of $(x_0, y_0)$ and that $\nabla f(x_0, y_0) = 0$. Set

$$A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \quad B = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0), \quad C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

and form the **discriminant** $D = AC - B^2$

1. If $D < 0$, then $(x_0, y_0)$ is a saddle point.
2. If $D > 0$, then $f$ takes on
   a local minimum at $(x_0, y_0)$ if $A > 0$,
   a local maximum at $(x_0, y_0)$ if $A < 0$. 

![Figure 15.5.5](image-url)
Lecture 08-1

4
5
B
B