15.4 - The Gradient as a Normal; Tangent Lines and Tangent Planes

At each point of the domain, the gradient vector $\nabla f$ (if $\neq 0$) is perpendicular to the level curve of $f$ that passes through that point.

The vector

$$ \mathbf{t}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) \mathbf{i} - \frac{\partial f}{\partial x}(x_0, y_0) \mathbf{j} $$

Is perpendicular to the gradient so it is the tangent vector.

The equation of the tangent line is:

$$ \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = 0 $$

And the equation of the normal line is:

$$ \frac{\partial f}{\partial y}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial x}(x_0, y_0)(y - y_0) = 0 $$
For \( f(x, y, z) \), the equation for the **tangent plane** is

\[
f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0
\]

And the **normal line** is

\[
x = x_0 + \frac{\partial f}{\partial x}(x_0, y_0, z_0) t,
\]

\[
y = y_0 + \frac{\partial f}{\partial y}(x_0, y_0, z_0) t,
\]

\[
z = z_0 + \frac{\partial f}{\partial z}(x_0, y_0, z_0) t.
\]

**Example:**

Find an equation for the tangent plane and scalar parametric equations for the normal line at the point \( P \).

\[
x^3 + y^3 = 3xyz, \quad P\left(1, 2, \frac{3}{2}\right)
\]
More examples:

Find the point(s) on the surface at which the tangent plane is horizontal.

\[ z - 2x^2 - 2xy + y^2 + 5x - 3y + 2 = 0 \]
Find the point(s) on the surface at which the tangent plane is horizontal.

\[ f(x, y) = 2x^2 + 2xy - y^2 - 5x + 3y - 2 \]
15.5 - Local Extreme Values

**DEFINITION 15.5.1 LOCAL MAXIMUM AND LOCAL MINIMUM**

Let \( f \) be a function of several variables and let \( x_0 \) be an interior point of the domain:

\( f \) is said to take on a *local maximum* at \( x_0 \) if

\[
    f(x_0) \geq f(x) \quad \text{for all } x \text{ in some neighborhood of } x_0;
\]

\( f \) is said to take on a *local minimum* at \( x_0 \) if

\[
    f(x_0) \leq f(x) \quad \text{for all } x \text{ in some neighborhood of } x_0.
\]

**THEOREM 15.5.2**

If \( f \) takes on a local extreme value at \( x_0 \), then

either \( \nabla f(x_0) = 0 \) or \( \nabla f(x_0) \) does not exist.

Critical points at which the gradient is zero are called *stationary points*. The stationary points that do not give rise to local extreme values are called *saddle points*. 
THEOREM 15.5.3  THE SECOND-PARTIALS TEST

Suppose that \( f \) has continuous second-order partial derivatives in a neighborhood of \((x_0, y_0)\) and that \( \nabla f(x_0, y_0) = 0 \). Set

\[
A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \quad B = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0), \quad C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)
\]

and form the discriminant \( D = AC - B^2 \)

1. If \( D < 0 \), then \((x_0, y_0)\) is a saddle point.
2. If \( D > 0 \), then \( f \) takes on
   
   a local minimum at \((x_0, y_0)\) if \( A > 0 \),
   
   a local maximum at \((x_0, y_0)\) if \( A < 0 \).

![Figure 15.5.5](image-url)
Examples:
Find the stationary points and the local extreme values.

1. \( f(x, y) = x^2 + 2xy + 3y^2 + 2x + 10y + 1 \)
2. \( f(x, y) = (x - y)(xy - 1) \)
15.6 - Absolute Extreme Values

How did we find absolute extreme values in Calc 1?

DEFINITION 15.6.1 ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM
Let $f$ be a function of several variables with domain $D$:

$f$ takes on an absolute maximum at $x_0$ if

$$f(x_0) \geq f(x) \quad \text{for all } x \in D;$$

$f$ takes on an absolute minimum at $x_0$ if

$$f(x_0) \leq f(x) \quad \text{for all } x \in D.$$

DEFINITION 15.6.2 BOUNDED SET
A set $S$ (of the real line, the plane, or three-space) is bounded if there exists a positive number $R$ such that

$$||x|| \leq R \quad \text{for all } x \in S.$$

THEOREM 15.6.3 EXTREME-VALUE THEOREM
If $f$ is continuous on a bounded closed set $D$, then $f$ takes on an absolute maximum value and an absolute minimum value.

Here is what we do:

1. Find the critical points in the interior of $D$.
2. Find the extreme points on the boundary of $D$.
3. Evaluate $f$ at the points found in Steps 1 and 2.
4. The largest of the numbers found in Step 3 is the absolute maximum value of $f$ and the smallest is the absolute minimum.
Examples:

1. Find the absolute extreme values taken on by $f$ on the set $D$.

$$f(x,y) = x^2 + y^2 + 3xy + 2, \quad D = \{(x,y) : x^2 + y^2 \leq 4\}.$$
Find the absolute extreme values of the function \( f(x, y) = 4xy - x^2 - y^2 - 6x \) on the triangular region \( D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3x\} \).
3. Find the absolute extreme values taken on by $f$ on the set $D$, where

$$f(x, y) = 9 - 8x + y + xy$$

and $D$ is the region bounded by $y = x^2$ and $y = 1$. 
15.7 - Maxima and Minima with Side Conditions

In section 15.6 we found absolute extrema of a function on a region that contained its boundary (like we did in Calc 1 when we used an interval). Now we will look at a different process – we will be optimizing a function $f$ subject to a constraint $g = 0$.

**Method of Lagrange:**

( $\lambda$ is called the *Lagrange multiplier*)

Set your constraint function = 0 (that’s $g$)

Then solve the system:

\[
\begin{align*}
\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\
g(x, y, z) &= 0
\end{align*}
\]

Last, plug in all solutions $(x, y, z)$ into $f$ to identify max and min values (if they exist).

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**Examples:**

1. Maximize $xy$ on the ellipse $4x^2 + 9y^2 = 36$. 

2. Minimize $x + 2y + 4z$ on the sphere $x^2 + y^2 + z^2 = 7$. 
3. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(3, 1, 3)$. 
15.8 – Differentials

Recall from Calc I:

Now, for \( f(x, y) \), the differential \( df = \nabla f(x) \cdot h \) or

\[
df = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y.
\]

And for \( f(x, y, z) \)

\[
df = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial z} \Delta z.
\]

Examples:
1. Find the differential \( df \)
   a. \( f(x, y, z) = xy + yz + xz \).

   \[
   \text{b. } f(x, y) = \sin(x + y) + \sin(x - y).
   \]
2. Calculate $\Delta u$ and $du$ for $u = x^2 - 3xy + 2y^2$ at $x = 2$, $y = -3$, $\Delta x = -0.3$, $\Delta y = 0.2$.

3. Use differentials to approximate $\sqrt{125\sqrt{17}}$.
15.9 - Reconstructing a Function from its Gradient

The gradient of a function $f(x, y)$ is $\nabla f(x, y) =$

So if we have a (gradient) vector in the form

$$P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$$

We can begin reconstruct the function by setting $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$

Now when integrating $\frac{\partial f}{\partial x}$ with respect to $x$, remember that $y$ is a constant!

Example: $xy^2 \mathbf{i} + x^2 y \mathbf{j}$
Note: Sometimes the given vector could not be the gradient of a function. Can you think of a quick way to determine if a vector is a gradient?

More examples:

2. \((y^3 + x)i + (x^2 + y)j\)

3. \((y^2e^x - y)i + (2ye^x - x)j\)
4. \((e^x + 2xy)i + (x^2 + \sin y)j\)

5. \((y^3 + x)i + (x^2 + y^2)j\)
6. \((\cos x - y \sin x)\mathbf{i} + (\cos x)\mathbf{j}\)

7. \((1 + e^x)\mathbf{i} + (xe^x + y^2)\mathbf{j}\)