14.2 – Gradients and Directional Derivatives

Properties of gradients:
\[ \nabla [f(x) + g(x)] = \nabla f(x) + \nabla g(x), \]
\[ \nabla [\alpha f(x)] = \alpha \nabla f(x), \]
\[ \nabla [f(x)g(x)] = f(x) \nabla g(x) + g(x) \nabla f(x). \]

Directional Derivatives:
\[ f'_u \] gives the directional derivative of \( f \) in the direction \( u \). In other words, \( f'_u \) gives the rate of change of \( f \) in the direction of \( u \).

\[ f'_u(x) = \nabla f(x) \cdot u \]

Example:
1. Find the directional derivative at the point \( P \) in the direction indicated.

\[ f(x, y) = x^2 + 3y^2 \] at \( P(1,1) \) in the direction of \( \mathbf{i} - \mathbf{j} \)
2. Find the directional derivative for \( f(x, y) = x^2 + 3y^2 \) at \( Q \left( -1, \frac{1}{2} \right) \) towards \( R \left( 2, \frac{3}{2} \right) \).

Note that the directional derivative in a direction \( u \) is the component of the gradient vector in that direction.

\[
f_u'(x) = \text{comp}_u \nabla f(x)
\]
Important:

from each point \( x \) of the domain, a differentiable function \( f \) increases most rapidly in the direction of the gradient (the rate of change at \( x \) being \( ||\nabla f(x)|| \)); the function decreases most rapidly in the opposite direction (the rate of change at \( x \) being \(-||\nabla f(x)|| \)).

Example:

Find a unit vector in the direction in which \( f \) increases most rapidly at \( P \) and give the rate of change of \( f \) in that direction; find a unit vector in the direction in which \( f \) decreases most rapidly at \( P \) and give the rate of change of \( f \) in that direction.

\[
f(x, y) = y^2e^{2x} \text{ at } P(0,1)
\]
14.3 – The Mean Value Theorem; Chain Rules

What was the MVT for functions of one variable?

**THEOREM 15.3.1 THE MEAN-VALUE THEOREM (SEVERAL VARIABLES)**

If \( f \) is differentiable at each point of the line segment \( \overline{ab} \), then there exists on that line segment a point \( c \) between \( a \) and \( b \) such that

\[
f(b) - f(a) = \nabla f(c) \cdot (b - a).
\]

Example: Let \( f(x, y) = x^3 - xy \) and let \( a = (0, 1) \) and \( b = (1, 3) \). Find a point \( c \) on the line segment \( \overline{ab} \) for which the mean value theorem (for several variables) is satisfied.
**THEOREM 15.3.4  CHAIN RULE (ALONG A CURVE)**

If $f$ is continuously differentiable on an open set $U$ and $\mathbf{r} = \mathbf{r}(t)$ is a differentiable curve that lies in $U$, then the composition $f \circ \mathbf{r}$ is differentiable and

$$
\frac{d}{dt}[f(\mathbf{r}(t))] = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).
$$

**Examples:**

1. Find $\frac{d}{dt}[f(\mathbf{r}(t))]$ given $f(x, y) = 6x + y$, $\mathbf{r}(t) = \sqrt{t} \mathbf{i} + 7t \mathbf{j}$
2. Find the rate of change of $f$ with respect to $t$ along the given curve.

$$f(x, y) = x^2 y, \quad r(t) = e^t \mathbf{i} + e^{-t} \mathbf{j}$$

3. Find the rate of change of $f$ with respect to $t$ along the given curve.

$$f(x, y, z) = x y - 2 y z$$
$$r(t) = 4 t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$$
Other chain rules:

If
\[ u = u(x, y) \quad \text{where} \quad x = x(s, t) \quad \text{and} \quad y = y(s, t) \]

Then
\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}.
\]

Example:

4. \( u = x^2 - 3xy + 2y^2 \quad x(t) = \cos t \quad y(t) = \sin t \). Find \( \frac{du}{dt} \).
And if

\[ u = u(x, y, z) \quad \text{where} \quad x = x(s, t) \quad y = y(s, t), \quad z = z(s, t) \]

Then

\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}, \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}.
\]

Example:

5. \( u = x^2 - xy + z^2 \quad x = s \cos t \quad y = \sin(t - s) \quad z = t \sin s \). Find \( \frac{du}{dt} \)
Suppose that $f(x, y)$ is a non-constant function that is continuously differentiable. That means $f$ is differentiable and its gradient $\nabla f$ is continuous. We saw last class that at each point in the domain $\nabla f$ (if $\neq 0$) points in the direction of the most rapid increase of $f$. Also, at each point of the domain, the gradient vector $\nabla f$ (if $\neq 0$) is perpendicular to the level curve of $f$ that passes through that point.

The vector

$$t(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)i - \frac{\partial f}{\partial x}(x_0, y_0)j$$

Is perpendicular to the gradient so it is the **tangent vector**.

The equation of the **tangent line** is:

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = 0$$

And the equation of the **normal line** is:

$$\frac{\partial f}{\partial y}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial x}(x_0, y_0)(y - y_0) = 0$$
Example:
Write an equation for the tangent line and an equation for the normal line at point $P$.

$$x^2 + xy + y^2 = 7 \quad P(1, -3)$$
For \( f(x, y, z) \), the equation for the tangent plane is

\[
f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0
\]

And the normal line is

\[
\begin{align*}
x &= x_0 + \frac{\partial f}{\partial x}(x_0, y_0, z_0)t, \\
y &= y_0 + \frac{\partial f}{\partial y}(x_0, y_0, z_0)t, \\
z &= z_0 + \frac{\partial f}{\partial z}(x_0, y_0, z_0)t.
\end{align*}
\]

Example:
Find an equation for the tangent plane and scalar parametric equations for the normal line at the point \( P \).

\[
x^3 + y^3 = 3xyz, \quad P\left(1, 2, \frac{3}{2}\right)
\]
More examples:
Find the point(s) on the surface at which the tangent plane is horizontal.

\[ z - 2x^2 - 2xy + y^2 + 5x - 3y + 2 = 0 \]
Find the point(s) on the surface at which the tangent plane is horizontal.

\[ f(x, y) = 2x^2 + 2xy - y^2 - 5x + 3y - 2 \]