- Test 1 begins next Saturday. That is 3/02.

- Practice Test 1 is now available. This is a Quiz grade!

- I will post a review sheet under 2/17 on the Calendar.

- I will add 5% of your best Practice Test 1 score to Test 1, if you score ≥70% on PT1 and do so before taking Test 1.

- We will review during lecture next Wednesday!
When you email me, you **MUST** include the following information:

- MATH 2433 Section 12242 and a searchable description in the **Subject Line**

- Your NAME and ID# in the **Body**

- Complete sentences, punctuation, and paragraph breaks
13.3 – Limits and Continuity

A set is **closed** if it contains ALL of its boundary points.

A set is **open** if it contains NONE of its boundary points.

A set **$S$** is a set

**Neighborhood of a point $c$** is

Point $b$ is a **Boundary Point of $S$** since ANY neighborhood has points inside and outside of $S$.

Point $a$ is an **Interior Point of $S$** since SOME neighborhood has only points in $S$.

**Boundary Set of $S$** is denoted "∂$S"

$S = (0, 1]$

$\partial S = \{0, 1\}$ is a subset

$(0-e, 0+\varepsilon)$

$(\frac{1}{2} - \frac{1}{4}, \frac{1}{2} + \frac{1}{4}) \subseteq S$, $\frac{1}{2}$ is an interior pt.
Examples:
1. \( \{(x,y) : 2 < x < 5, 4 < y < 6\} = S \)

\[ S = \{(x,y) | 2 \leq x \leq 5, \ y = 4 \text{ or } 6\} \cup \{(x,y) | x = 2 \text{ or } 5, \ 4 \leq y \leq 6\} \]

\( S \) is open
2. \( \{(x, y): 2 \leq x \leq 5, 4 \leq y \leq 6\} = S \)

\[ S \]

\( S \) is the same as \# 1

Also, \( S \) is contained in \( S \) here

\( S \subseteq S \)

Here \( S \) is \boxed{\text{closed}}

\( \partial S = \{ (x, y) | 2 \leq x \leq 5, y = 4 \text{ or } 6 \} \leftarrow \text{Two horizontal segments} \)

\( \cup \{ (x, y) | x = 2 \text{ or } 5, 4 \leq y \leq 6 \} \leftarrow \text{Two vertical segments} \)
3. \( \{(x, y) : 2 \leq x \leq 5, 4 < y < 6\} = S \)

\( (3, 6) \)

\( (5, 5) \)

\( y = 6 \)

\( y = 4 \)

\( x = 2 \)

\( x = 5 \)

\( S \) is as in #1

\( S \) is not open since \((5, 5) \in S\)

\( S \) is not closed since \((3, 6) \notin S\)
4. \( S = \{(x, y) : y \leq x^2\} \)

\( S = \{ (x, y) \mid y = x^2 \} \)

and \( \mathcal{C} S \leq S \)

\( S \) is closed
5. \( S = \{(x, y, z) : x^2 + y^2 \leq 1, z > 1\} \)

\( S \) is an infinitely tall cylinder w/ radius = 1 and no bottom.

Why is \( S \) not open?

\((1, 0, 2) \in S\)

Why is \( S \) not closed?

\((0, 0, 1) \notin S\)

\( \partial S = \{(x, y, z) | x^2 + y^2 \leq 1, z = 1\} \) \( \quad \text{Bottom} \)

\( \cup \{(x, y, z) | x^2 + y^2 = 1, z \geq 1\} \) \( \quad \text{Cylindrical Side} \)
1. For a set to be closed it must
   a. Contain all of its boundary points
   b. Have a limit on its domain ← nonsense
   c. Contain some of its boundary points ← not open
   d. Not contain any boundary points ← open
   e. None of these

2. Give the boundary of the set $S = \{(x, y): 1 < x^2 + y^2 \leq 4\}$
   a. the circle $x^2 + y^2 = 4$
   b. the circle $x^2 + y^2 = 1$
   c. this set has no boundary ← incorrect
   d. the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$
   e. none of these
**Limits**

In calculus 1, we said \( \lim_{x \to a} f(x) \) exists iff

\[
\lim_{x \to a} f(x) = L
\]

And we learned the *Formal Definition of Limit*:

The limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \)

\[\lim_{x \to a} f(x) = L\]

if and only if, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( 0 < |x - a| < \delta \) implies that \( |f(x) - L| < \varepsilon \).

Both exist and agree.

\( x \neq a \)

\( x \) is close to \( a \)

A deleted neighborhood of \( a \)

The reader’s ability to resolve output difference
Now, in $\mathbb{R}^2$ and $\mathbb{R}^3$:

**DEFINITION 14.6.1 THE LIMIT OF A FUNCTION OF SEVERAL VARIABLES**

Let $f$ be a function defined at least on some deleted neighborhood of $x_0$.

\[
\lim_{x \to x_0} f(x) = L
\]

if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

\[
0 < ||x - x_0|| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon.
\]

Do all paths yield the same limit value???

There are infinitely many paths through any point $x_0$.

If any two paths yield different limit values

then $\lim_{x \to x_0} f(x)$ DNE.
Note:

\[ \lim_{x \to a} = \text{Where is the function of } x? \]

Never has a meaning!
Examples:

1. Find \( \lim_{(x,y) \to (0,0)} f(x,y) \) for \( f(x,y) = \frac{y^2}{x^2 + y^2} \)

\[
\begin{align*}
\lim_{(x,y) \to (0,0)} f(x,y) &= \lim_{(x,y) \to (0,0)} \frac{y^2}{x^2 + y^2} \\
&= \lim_{y \to 0} \frac{y^2}{y^2} = 1 \\
&= \begin{cases} 
1 & \text{if } y \neq 0 \\
\text{DNE if } y = 0
\end{cases}
\end{align*}
\]

Note: \( \frac{y^2}{y^2} \neq 1 \)

2. \( y \)-axis, \( x = 0 \)

\[
\lim_{y \to 0} f(0,y) = \lim_{y \to 0} \frac{y^2}{x^2 + y^2} = 1 \\
\text{if } y \neq 0
\]

3. \( y = x^2 \)

\[
\lim_{x \to 0} f(x,x^2) = \lim_{x \to 0} \frac{(x^2)^2}{x^2 + (x^2)^2} = \lim_{x \to 0} \frac{x^4}{x^2 + x^4} = \lim_{x \to 0} \frac{4x^3}{2x + 4x^3} = \lim_{x \to 0} \frac{12x^2}{2 + 12x^2} = 0
\]

\[
\lim_{x \to 0} f(x,0) = \lim_{x \to 0} \frac{0}{x^2} = 0
\]

4. \( y = x, f(x,x) \)

\( f(0,0) \) is undefined

I suspect this limit DNE.

Let's show this using paths (curves) through \( (0,0) \).

Examples:

- \( x \)-axis, \( y = 0 \)
  Look at \( f(x,0) \)
- \( y \)-axis, \( x = 0 \)
  Look at \( f(0,y) \)
- \( y = mx \)
  Look at \( f(x,mx) \)
- \( y = x^2, f(x,x^2) \)
- \( y = \sin(x), f(x,\sin(x)) \)

And infinitely many more
2. Find \( \lim_{(x,y) \to (1,2)} f(x,y) \) for \( f(x,y) = \frac{y^2}{x^2 + y^2} \)

This function is continuous except at \((x,y) = (0,0)\).

Thus, \( \lim_{(x,y) \to (1,2)} f(x,y) = f(1,2) = \frac{4}{1+4} = \frac{4}{5} \).
3. Show that the function \( f(x, y) = \frac{x^2 y}{x^4 + y^2} \) has a limit as \((x, y) \to (0, 0)\) along any line through the origin, but
\[
\lim_{(x, y) \to (0, 0)} f(x, y)
\]
still does not exist.

\[
y = mx \quad \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x^2 \cdot mx}{x^4 + m^2 x^2} = \lim_{x \to 0} \frac{x \cdot mx}{x(x^2 + m^2)} = 0
\]

\[
y = ax^2, x = 0 \quad \lim_{y \to 0} f(0, y) = \lim_{y \to 0} \frac{0 \cdot x^2}{y^4 + y^2} = \lim_{y \to 0} \frac{0}{y^2} = 0
\]

Thus, for any line through \((0, 0)\), \( \lim_{(x, y) \to (0, 0)} f(x, y) = 0 \)

Check \( y = x^2 \quad \lim_{x \to 0} f(x, x^2) = \lim_{x \to 0} \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2} \)

Since one path gives 0 and another gives \( \frac{1}{2} \)

**We cannot define \( \lim_{(x, y) \to (0, 0)} f(x, y) \) so it DNE**
Lec Pop 06-1

3 C
4 B
5 E