- Test 1 registration opens 12:00 a.m. on 9/21
- We will spend time reviewing in lecture on one of 10/02 or 10/04.
13.4: Arc Length
Recall from calc II: Arc length and speed:

\[ L(c) = \int_a^b \sqrt{\left[ x'(t) \right]^2 + \left[ y'(t) \right]^2} \, dt \]

When \( x = x(t), y = y(t), t \in [a,b] \)

\[ L(c) = \int_a^b \sqrt{1 + \left[ f'(x) \right]^2} \, dx \]

Let \( x(t) = t, y(t) = f(t), t \in [a,b] \)
\( x'(t) = 1, y'(t) = f'(t) \)

When \( y = f(x), x \in [a,b] \)

\[ L(c) = \int_a^\beta \sqrt{\left[ \rho(\theta) \right]^2 + \left[ \rho'(\theta) \right]^2} \, d\theta \]

When we have the polar curve \( r = \rho(\theta), \theta \in [a,\beta] \)
\( x = r \cos \theta = \rho(\theta) \cos \theta, y = r \sin \theta = \rho(\theta) \sin \theta \)

Check that: \( [x'(\theta)]^2 + [y'(\theta)]^2 = [\rho(\theta)]^2 + [\rho'(\theta)]^2 \)
Speed along a curve:

Path \((x(t), y(t))\) takes from 0 to \(t\): 
\[ s = \int_{0}^{t} \sqrt{[x'(u)]^2 + [y'(u)]^2} \, du \]

So, 
\[ s'(t) = \frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2} \]

\[ \text{speed at time } t \]

When \( \mathbf{r}(t) = (x(t), y(t)) \)

speed of this point along the curve is
\[ \frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2} \]
Now apply this to a path $C$ traced out by $x = x(t)$, $y = y(t)$ and $z = z(t)$ for $t \in [a, b]$ and we have:

$$L(C) = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2} \, dt$$

Or

$$L(C) = \int_a^b \|r'(t)\| \, dt$$

So, $\|r'(t)\|$ is also the speed.
Example:
Find the length of the curve given by

\[
r(t) = 2t \mathbf{i} + \sqrt{5} \ln(\sec t) \mathbf{j} + t \mathbf{k} \quad \text{from } t = 0 \text{ to } \frac{\pi}{4}
\]

\[
\begin{align*}
\overrightarrow{r}'(t) &= \left( 2, \sqrt{5} \frac{\sec t}{\sec t \cdot \sec t \cdot \tan t}, 1 \right) = \left( 2, \sqrt{5} \tan t, 1 \right) \\
\left\| \overrightarrow{r}'(t) \right\| &= \sqrt{2^2 + 5\tan^2 t + 1^2} = \sqrt{5 + 5\tan^2 t} = \sqrt{5(1 + \tan^2 t)} \\
&= \sqrt{5\sec^2 t} = \sqrt{5} \left| \sec t \right| = \sqrt{5} \sec t \quad t \in \left[0, \frac{\pi}{4}\right]
\end{align*}
\]

\[
\ell(c) = \int_{0}^{\frac{\pi}{4}} \sqrt{5} \sec t \, dt = \sqrt{5} \left. \ln \left| \sec t + \tan t \right| \right|_{0}^{\frac{\pi}{4}}
\]

\[
= \sqrt{5} \left[ \ln \left( \sec \left( \frac{\pi}{4} \right) + \tan \left( \frac{\pi}{4} \right) \right) - \ln \left( \sec (0) + \tan (0) \right) \right] \\
= \sqrt{5} \left[ \ln \left( \sqrt{2} + 1 \right) - \ln \left( 1 + 0 \right) \right] \\
= \sqrt{5} \ln \left( \sqrt{2} + 1 \right)
\]
Speed:
Let $\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$, $t \in [a, b]$ be a continuously differentiable curve. If $s$ is the length of the curve from the tip of $\mathbf{r}(a)$ to the tip of $\mathbf{r}(t)$, then the speed is:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

**Speed is**

$$\frac{ds}{dt} = \left\| \mathbf{r}'(t) \right\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$
Note: a curve is said to be a unit speed curve if \( \mathbf{r}'(t) \) is a unit vector for all \( t \).

**13.5 Curvilinear Motion; Curvature**

We can describe the position of a moving object at time \( t \) by a radius vector \( \mathbf{r}(t) \). As \( t \) ranges over a time interval \( I \), the object traces out some path \( C : \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}, \ t \in I \).

If \( \mathbf{r} \) is twice differentiable, then we can find

\[
\mathbf{r}'(t) = \mathbf{v}(t) - \text{velocity} \quad \text{and} \quad \mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t) - \text{acceleration}
\]

Also,

\[
\| \mathbf{v}(t) \| = \| \mathbf{r}'(t) \| \text{ is the speed at time } t.
\]

The magnitude of the velocity vector is thus the rate of change of arc distance with respect to time. This is why we call it the speed of the object.

Velocity is speed with direction

that is \( \mathbf{v}(t) = \| \mathbf{v}(t) \| \mathbf{T}(t) \),

\( \mathbf{T}(t) \) speed

\( \mathbf{v}(t) \) direction.
1. Which of the following is the speed of a curve given by $\mathbf{r}(t)$?
   a. $\frac{ds}{dt}$
   b. $\|\mathbf{r}'(t)\|$
   c. $\mathbf{r}'(t)$
   d. both a and b
   e. a, b and c
   f. none of these
The Curvature of a Plane Curve

In this figure we have a curve through point $P$ with tangent line $l$ that intersects the $x$-axis at angle $\phi$

Then $\kappa = \left| \frac{d\phi}{ds} \right|$ (the magnitude of the rate of change of the angle per unit of arc length) is called the curvature.

If we have the tangent vector $\mathbf{T}$, then $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$

\[
\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} \right\|
\]

\[
= \frac{\left\| \mathbf{T}'(t) \right\|}{\left\| \mathbf{T}'(t) \right\|} = \frac{\left\| \mathbf{T}'(t) \right\|}{\left\| \mathbf{T}'(t) \right\|}
\]

much easier to find!
These are the 2-D special cases of $\kappa = \frac{\|T''(t)\|}{\|T'(t)\|}$

If a curve is given by $y = y(x)$, then

$$\kappa = \frac{|y''|}{\sqrt{1 + (y')^2}}^{3/2} \quad (1)$$

Now, if we have a curve given parametrically, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ and

$$\kappa = \frac{|x'y'' - y'x''|}{\sqrt{(x'')^2 + (y'')^2}}^{3/2} \quad (2)$$

To see (1) in terms of (2),

If $y = y(x)$ then $x' = 1$, $y' = y'$,

$$x'' = 0, \quad y'' = y''$$

$$|x'y'' - y'x''| = |1 \cdot y'' - y' \cdot 0| = |y''|$$

and $[|x'|^2 + |y'|^2]^{3/2} = [1 + (y')^2]^{3/2}$
Note:

- Along a straight line, the curvature is constantly zero.
- Along a circle of radius $r$ the curvature is constantly $1/r$.

Also, the reciprocal of the curvature is called the \textit{radius of curvature}:

$$\rho = \frac{1}{\kappa}$$

And the point at the distance $\rho$ from the curve in the direction of the principal normal is called the \textit{center of curvature}.
Examples:

1. Find the curvature of the given curve: \( y = \ln(\sec(x)) \)

\[ k = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}} \]

\[ y' = \frac{1}{\sec(x)} \cdot \sec(x) \tan(x) = \tan(x) \]

\[ y'' = \sec^2(x) \]

\[ k = \frac{\sec^2(x)}{\left[1 + (\tan(x))^2\right]^{3/2}} = \frac{\sec^2(x)}{\sec^2(x)}^{3/2} = \frac{\sec^2(x)}{\sec(x)^3} = \frac{1}{\sec(x)} = |\cos(x)| \]

\[ k = |\cos(x)| \]
2. Express the curvature in terms of $t$.

$$\mathbf{r}(t) = 2t \mathbf{i} + 3t^3 \mathbf{j}$$

\[
x = 2t, \quad y = 3t^3 \\
x' = 2, \quad y' = 9t^2 \\
x'' = 0, \quad y'' = 18t
\]

\[
\kappa = \frac{|x'y'' - y'x''|}{\left[ (x')^2 + (y')^2 \right]^{3/2}} = \frac{18t}{\left[ 4 + 81t^4 \right]^{3/2}}
\]
The Curvature of a Space Curve
A space curve bends in two ways. It bends in the osculating plane (the plane of the unit tangent $\mathbf{T}$ and the principal normal $\mathbf{N}$) and it bends away from that plane. The first form of bending is measured by the rate at which the unit tangent $\mathbf{T}$ changes direction. The second form of bending is measured by the rate at which the vector $\mathbf{T} \times \mathbf{N}$ changes direction. We will concentrate here on the first form of bending, the bending in the osculating plane. The measure of this is called curvature.

If the space curve is given in terms of a parameter $t$, $C : \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}, \ t \in [a,b]$ then the curvature is:

$$\kappa = \frac{\left\| \frac{d\mathbf{T}}{dt} \right\|}{ds/dt} = \frac{\left\| \frac{\mathbf{r}''(t)}{\left(\mathbf{r}'(t)\right)'} \right\|}{\left\| \mathbf{r}'(t) \right\|}$$
Example: Calculate the curvature of \( \mathbf{r}(t) = \cos t \mathbf{i} + t \mathbf{j} + \sin t \mathbf{k} \)

\[
\mathbf{r}'(t) = \left( -\sin t, 1, \cos t \right), \quad \| \mathbf{r}'(t) \| = \sqrt{\sin^2 t + 1 + \cos^2 t} = \sqrt{2}
\]

\[
\mathbf{T}(t) = \frac{1}{\| \mathbf{r}'(t) \|} \mathbf{r}'(t) = \frac{1}{\sqrt{2}} \left( -\sin t, 1, \cos t \right)
\]

\[
\mathbf{T}''(t) = \frac{1}{\sqrt{2}} \left( -\cos t, 0, -\sin t \right)
\]

\[
\| \mathbf{T}''(t) \| = \left| \frac{1}{\sqrt{2}} \sqrt{\cos^2 t + 0 + \sin^2 t} \right| = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}
\]

\[
\kappa = \frac{\| \mathbf{T}''(t) \|}{\| \mathbf{r}'(t) \|} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{1}{2}
\]
Components of Acceleration

Given \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \Rightarrow \mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}}{ds/dt} \), so:

\[
\mathbf{v} = \frac{ds}{dt} \cdot \mathbf{T}(t)
\]

\[
\mathbf{v}'(t) = \frac{ds}{dt} \cdot \mathbf{T}'(t) + \frac{d}{dt} \left( \frac{ds}{dt} \right) \mathbf{T}(t)
\]

This means that \( \mathbf{a} = \mathbf{v}'(t) = \mathbf{T}'(t) \frac{ds}{dt} + \mathbf{T}(t) \frac{d}{dt} \left( \frac{ds}{dt} \right) \) and since

\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \Rightarrow \mathbf{T}'(t) = \mathbf{N}(t) \|\mathbf{T}'(t)\|
\]

\[
\mathbf{a}(t) = \mathbf{N}(t) \|\mathbf{T}'(t)\| \frac{ds}{dt} + \mathbf{T}(t) \frac{d^2 s}{dt^2} = \mathbf{N}(t) \cdot \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} + \mathbf{T}(t) \cdot \frac{d}{dt} \left( \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\|} \right)
\]

So,

\[
\mathbf{a}(t) = \mathbf{N}(t) \|\mathbf{T}'(t)\| \frac{ds}{dt} + \mathbf{T}(t) \frac{d^2 s}{dt^2}
\]

\[
\mathbf{a}(t) = \mathbf{N}(t) \cdot \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} + \mathbf{T}(t) \cdot \frac{d}{dt} \left( \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\|} \right)
\]

\[
\mathbf{a}(t) = \kappa \cdot \|\mathbf{T}'(t)\| \mathbf{N}(t) + \frac{d}{dt} \left( \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\|} \right) \mathbf{T}(t)
\]

\( \frac{ds}{dt} \) Change in direction of \( \mathbf{v}(t) \)

\( \frac{d}{dt} \) Change in size of \( \mathbf{v}(t) \)
The acceleration vector lies in the osculating plane, the plane of $T$ and $N$. The tangential component of acceleration is given by:

$$a_T = \frac{d}{dt} \left( \| \vec{r}'(t) \| \right) = \frac{d^2 s}{dt^2}$$

And this depends only on the change of speed of the object; if the speed is constant, the tangential component of acceleration is zero and the acceleration is directed entirely toward the center of curvature of the path.

The normal component of acceleration is

$$a_N = \kappa \cdot \| \vec{r}'(t) \|^2$$

And depends both on the speed of the object and the curvature of the path. At a point where the curvature is zero, the normal component of acceleration is zero and the acceleration is directed entirely along the path of motion. If the curvature is not zero, then the normal component of acceleration is a multiple of the square of the speed.
If we take the dot product of \( \mathbf{T} \) with \( \mathbf{a} \), we have

\[
\mathbf{T} \cdot \mathbf{a} = a_T (\mathbf{T} \cdot \mathbf{T}) + a_N (\mathbf{T} \cdot \mathbf{N}) = a_T
\]

Which means we can write:

\[
\kappa = \frac{\| \mathbf{v} \times \mathbf{a} \|}{(ds/dt)^3} = \frac{\| \mathbf{v} \times \mathbf{a} \|}{\| \mathbf{v} \|^3}
\]
Summary of $\kappa$ for Space Curves:

\[ \kappa = \frac{||\vec{T}''(t)||}{||\vec{T}'(t)||} \]

\[ \kappa = \frac{||\vec{\nabla} \times \vec{a}||}{||\vec{\nabla}||^3} \]

When to use: If you already found $\vec{N}$

If you are finding $\kappa$ from scratch and $\vec{T}''(t)$ is difficult
Examples:

1. Interpret \( \mathbf{r}(t) \) as the position of a moving object at time \( t \). Determine the tangential and normal components of acceleration.

\[
\mathbf{r}(t) = \cos(t)\mathbf{i} + t\mathbf{j} + \sin(t)\mathbf{k}
\]

We already found \( \|\mathbf{r}'(t)\| = \sqrt{2}, \|\mathbf{r}''(t)\| = \frac{1}{\sqrt{2}}, \kappa = \frac{1}{2} \)

\[
\begin{align*}
\mathbf{a}_T &= \frac{d}{dt} \left( \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \right) = \frac{d}{dt} \left( \frac{\sqrt{2}}{2} \right) = \boxed{0} \\
\mathbf{a}_N &= \kappa \cdot \|\mathbf{r}'(t)\|^2 = \frac{1}{2} \cdot (\sqrt{2})^2 = \frac{\sqrt{2}}{2} = \boxed{1} \\
\end{align*}
\]

So, \( \mathbf{a}(t) = 0 \cdot \mathbf{r}'(t) + 1 \cdot \mathbf{N}(t) = \mathbf{N}(t) \)
2. The vector function \( \mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + \frac{\sqrt{3}}{2}t^2\mathbf{k} \) determines a curve \( C \) in space. Assume \( t > 0 \)

(a) Find the unit tangent vector \( \mathbf{T}(t) \) and the principal normal vector \( \mathbf{N} \).

\[
\mathbf{r}'(t) = (-\sin t + \sin t + t \cdot \cos t, \cos t - 1 \cdot \cos t + t (\sin t), \sqrt{3} t)
\]

\[
= (t \cos t, t \sin t, \sqrt{3} t) = t (\cos t, \sin t, \sqrt{3})
\]

\[
||\mathbf{r}'(t)|| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t + 3} = t \sqrt{4} = 2t
\]

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{1}{2t} (\cos t, \sin t, \sqrt{3}) = \frac{1}{2} (\cos t, \sin t, \sqrt{3})
\]

\[
\mathbf{T}'(t) = \frac{1}{2} (-\sin t, \cos t, 0), \quad ||\mathbf{T}'(t)|| = \frac{1}{2} \cdot 1 = \frac{1}{2}
\]

\[
\mathbf{N}(t) = \frac{1}{||\mathbf{T}'(t)||} \mathbf{T}'(t) = \frac{1}{\frac{1}{2}} (-\sin t, \cos t, 0) = (-\sin t, \cos t, 0)
\]
(b) Find the curvature, $\kappa$ of $C$.

$$\kappa = \frac{\| \vec{T}'(t) \|}{\| \vec{T}''(t) \|} = \frac{1}{2t} = \frac{1}{4t}$$

(c) Determine the tangential and normal components of acceleration; express the acceleration vector, $\mathbf{a}(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$.

$$a_T = \frac{d}{dt} \| \vec{T}'(t) \| = \frac{d}{dt} (2t) = 2$$

$$a_N = \kappa \cdot \| \vec{T}'(t) \|^2 = \frac{1}{4t} \cdot (2t)^2 = \frac{4t^2}{4t} = t$$

$$\vec{a}(t) = 2 \cdot \vec{T}'(t) + t \cdot \vec{N}(t)$$
Interpret $\mathbf{r}(t)$ as the position of a moving object at time $t$.

$$\mathbf{r}(t) = \cos(2t)i + \sin(2t)j + \sqrt{3}k$$

2. Find the unit tangent vector

   a. $T(t) = -\sin(2t)i + \cos(2t)j + \frac{\sqrt{3}}{2}k$
   b. $T(t) = -\sin(2t)i + \cos(2t)j$
   c. $T(t) = -\cos(2t)i - \sin(2t)j$

3. Find the principal normal vector

   a. $N(t) = -\sin(2t)i + \cos(2t)j + \frac{\sqrt{3}}{2}k$
   b. $N(t) = -\sin(2t)i + \cos(2t)j$
   c. $N(t) = -\cos(2t)i - \sin(2t)j$

4. Find the curvature

   a. $\kappa = 1$
   b. $\kappa = 2$
   c. $\kappa = 3$
In Chapter 13: \[ \vec{f}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \]
scalar input and vector output

In Chapter 14: \[ f(x,y) \text{ or } f(x,y,z) \]
vector input and scalar output
14.1 Examples of real-valued functions of two and three variables
(read this section!!)
Finding domain and range:

1. \( f(x, y) = \sqrt{xy} \)

\textit{Domain - Set of allowed input values}
\textit{Range - Set of possible output values}

Domain Rules:
1. \( \frac{1}{u} \Rightarrow u \neq 0 \)
2. \( \sqrt{u} \Rightarrow u \geq 0 \)
3. \( \log_a(u) \Rightarrow u > 0 \)
LeC Pop 04-1

5 E