12.7 Planes

Angle between two planes: \( \cos \theta = \left| \mathbf{u}_n \cdot \mathbf{u}_{n'} \right| \)

6) Find the angle between the planes. \( 2x - 2y + 2z + 5 = 0 \) and \( 2x - y + z - 3 = 0 \)

Distance from a point to a plane \( d(P, \phi) = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \)

7) Find the distance from the point \( P(1, -1, 2) \) to the plane \( 2x - y + 2z + 3 = 0 \)
8) Find an equation in $x, y, z$ for the plane that passes through the given points.

$P(1, -1, 2), Q(3, -1, 3), R(2, 0, 2)$
Two planes will be parallel if their normal vectors are scalar multiples of each other. If two planes are not parallel, then they will intersect in a line. The line of intersection will have a direction vector equal to the cross product of their norms.

9) Find a set of scalar parametric equations for the line formed by the two intersecting planes.

\[ x + y + z + 1 = 0 \quad \text{and} \quad 3x - 3y + 3z + 6 = 0 \]
13.1 Vector Functions

Vector functions in $\mathbb{R}^3$ are in the form: $\mathbf{f}(t) = f_1(t)i + f_2(t)j + f_3(t)k$ where $f_1, f_2,$ and $f_3$ are the component functions.

The scalar functions in the case of a line would take the form:

$$f_1(t) = x_0 + d_1t \quad f_2(t) = y_0 + d_2t \quad f_3(t) = z_0 + d_3t$$

We can think of the vector $f_1(t)i + f_2(t)j + f_3(t)k$ or $(f_1(t), f_2(t), f_3(t))$ as the position of a moving particle at time $t$. In order to sketch the graph of a vector function, it is easier to look at it in terms of the scalar vectors:

$$x(t) = f_1(t) \quad y(t) = f_2(t) \quad z(t) = f_3(t)$$

The domain of a vector function is the set of all $t$ values for which all the component functions are defined. It is the largest possible interval for which all three components are defined.

$$\lim_{t \to t_0} f(t) = \mathbf{L} \quad \text{iff} \quad \lim_{t \to t_0} f_1(t) = L_1, \lim_{t \to t_0} f_2(t) = L_2, \lim_{t \to t_0} f_3(t) = L_3$$

So, if we can find the limit, we can find the derivative:

A vector function $\mathbf{f}$ is differentiable at $t$ if

$$\lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h} \text{ exists.}$$

And if the limit exists then $\mathbf{f}'(t) = \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$

Or rather, given $\mathbf{f}(t) = f_1(t)i + f_2(t)j + f_3(t)k$, $\mathbf{f}'(t) = f'_1(t)i + f'_2(t)j + f'_3(t)k$

Note, if any component of $\mathbf{f}(t)$ is not differentiable, then $\mathbf{f}'(t)$ DNE.
Example: Find the derivative of
$$f(t) = e^{2t}i + \ln(3t)j + \arctan(3t)k$$

Integration:
$$\int_a^b f(t) \, dt = \left( \int_a^b f_1(t) \, dt \right) i + \left( \int_a^b f_2(t) \, dt \right) j + \left( \int_a^b f_3(t) \, dt \right) k$$

Example: Find $\int_0^1 f(t) \, dt$ for
$$f(t) = \left( \frac{3}{1+t^2} \right) i + \left( 2\sec^2(t) \right) j$$
13.2 Differentiation formulas

Properties of the derivatives:

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of $t$, and let $f$ be a differentiable real-valued function of $t$, and let $c$ be a scalar.

1. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

2. $\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

3. $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

4. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

5. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

6. $\frac{d}{dt}[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$

7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$

Example: If $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t \mathbf{k}$ and $\mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$, find $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$.
13.3 Curves

The parameterization of the curve represented by the vector-valued function
\[ r(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \]
is smooth on an open interval \( I \) if \( x'(t), y'(t), \) and \( z'(t) \) are continuous on \( I \) and \( r'(t) \neq 0 \) for any value of \( t \) on the interval \( I \).

Let \( C \) be a smooth curve represented by \( r \) on an open interval \( I \). The unit tangent vector \( T(t) \) at \( t \) is defined to be
\[ T(t) = \frac{r'(t)}{\|r'(t)\|}, r'(t) \neq 0 \]

A tangent line to a curve at a point is the line passing through the given point that is parallel to the tangent vector.

Examples:
Find the tangent vector \( r'(t) \) and the unit tangent vector \( T(t) \) at the point where \( t = \pi / 4 \) and parameterize the tangent line at the indicated point.
\[ r(t) = 2 \cos(t) \mathbf{i} + 2 \sin(t) \mathbf{j} + t \mathbf{k} \]
Now if $T'(t) = 0$ then the unit tangent vector does not change direction.
If $T'(t) \neq 0$, then we can find the **principal normal vector**:

$$N(t) = \frac{T'(t)}{||T'(t)||}$$

Example: Find the principle normal for $r(t) = 2\cos(t)i + 2\sin(t)j + tk$, $t = \frac{\pi}{4}$

The plane determined by the unit tangent vector and the principal normal vector is called the **osculating plane**.

Continuing the problem above:
Intersecting Curves:

Two curves

\[ \mathbf{r}_1(t) = x_1(t) \mathbf{i} + y_1(t) \mathbf{j} + z_1(t) \mathbf{k}, \quad \mathbf{r}_2(u) = x_2(u) \mathbf{i} + y_2(u) \mathbf{j} + z_2(u) \mathbf{k} \]

intersect iff there are numbers \( t \) and \( u \) for which

\[ \mathbf{r}_1(t) = \mathbf{r}_2(u). \]

If two curves \( C_1 \) and \( C_2 \) intersect at a point \( \mathbf{r}_1(t_1) = \mathbf{r}_2(u_2) \), we define the angle between the curves at this point to be the angle between the corresponding tangent vectors \( \mathbf{r}'_1(t_1) \) and \( \mathbf{r}'_2(u_2) \).

\[
\cos \theta = \frac{\mathbf{r}'_1(t_1) \cdot \mathbf{r}'_2(u_2)}{\Vert \mathbf{r}'_1(t_1) \Vert \Vert \mathbf{r}'_2(u_2) \Vert}
\]

Example:

Find the point at which the curves intersect and find the angle of intersection.

\[ \mathbf{r}_1(t) = e^{it} \mathbf{i} + 4 \sin(t + \pi/2) \mathbf{j} + (t^2 - 1) \mathbf{k} \]

\[ \mathbf{r}_2(u) = u \mathbf{i} + 4 \mathbf{j} + (u^2 - 2) \mathbf{k} \]
13.4 Arc Length
Recall from calc II: (9.8) Arc length and speed:

\[ L(c) = \int_a^b \sqrt{\left( x'(t) \right)^2 + \left( y'(t) \right)^2} \, dt \]

\[ L(c) = \int_a^b \sqrt{1 + \left( f'(x) \right)^2} \, dx \]

\[ L(c) = \int_a^\beta \sqrt{\left( \rho'(\theta) \right)^2 + \left( \rho(\theta) \right)^2} \, d\theta \]

Speed along a curve:
Path \((x(t), y(t))\) takes from 0 to \(t\):

\[ s = \int_0^t \sqrt{\left( x'(u) \right)^2 + \left( y'(u) \right)^2} \, du \]

So, \( v(t) = s'(t) = \sqrt{\left( x'(t) \right)^2 + \left( y'(t) \right)^2} \)

Now apply this to a path \(C\) traced out by \(x = x(t), y = y(t)\) and \(z = z(t)\) for \(t \in [a, b]\) and we have:

\[ L(C) = \int_a^b \sqrt{\left( x'(t) \right)^2 + \left( y'(t) \right)^2 + \left( z'(t) \right)^2} \, dt \]

Or

\[ L(C) = \int_a^b \| r'(t) \| \, dt \]

Example:
Find the length of the curve given by

\[ r(t) = 2t \hat{i} + \sqrt{5} \ln(\sec t) \hat{j} + t \hat{k} \quad \text{from} \quad t = 0 \text{ to } \frac{\pi}{4} \]
Speed:
Let \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}, \ t \in [a, b] \) be a continuously differentiable curve. If \( s \) is the length of the curve from the tip of \( \mathbf{r}(a) \) to the tip of \( \mathbf{r}(t) \), then the speed is:

\[
\frac{ds}{dt} = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2}
\]

Note: a curve is said to be a unit speed curve if \( \mathbf{r}'(t) \) is a unit vector for all \( t \)

### 13.5 Curvilinear Motion; Curvature
We can describe the position of a moving object at time \( t \) by a radius vector \( \mathbf{r}(t) \). As \( t \) ranges over a time interval \( I \), the object traces out some path \( C : \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}, \ t \in I \)

If \( \mathbf{r} \) is twice differentiable, then we can find

\[
\mathbf{r}'(t) = \mathbf{v}(t) \text{- velocity} \quad \text{and} \quad \mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t) \text{- acceleration}
\]

Also, \( \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| \) is the speed at time \( t \). The magnitude of the velocity vector is thus the rate of change of arc distance with respect to time. This is why we call it the speed of the object.

Example:
Sketch the curve. Then compute and sketch the acceleration vector at the indicated points.
\( \mathbf{r}(t) = t^3 \mathbf{i} + t \mathbf{j}, \ t \text{ real}; \) at \( t = -1/2, 1/2, 1 \)
The Curvature of a Plane Curve

In this figure we have a curve through point \( P \) with tangent line \( l \) that intersects the \( x \)-axis at angle \( \phi \).

Then \( \kappa = \left| \frac{d\phi}{ds} \right| \) (the magnitude of the rate of change of the angle per unit of arc length) is called the curvature.

If we have the tangent vector \( T \), then \( \kappa = \left\| \frac{dT}{ds} \right\| \).

If a curve is given by \( y = y(x) \), then \( \kappa = \frac{|y''|}{\sqrt{1 + (y')^2}}^{3/2} \).

Now, if we have a curve given parametrically, \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \) and

\[
\kappa = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}
\]

Note:
- Along a straight line, the curvature is constantly zero.
- Along a circle of radius \( r \) the curvature is constantly \( 1/r \).

Also, the reciprocal of the curvature is called the radius of curvature: \( \rho = \frac{1}{\kappa} \).
And the point at the distance \( \rho \) from the curve in the direction of the principal normal is called the center of curvature.
Examples:

1. Find the curvature of the given curve: \( y = \ln(\sec(x)) \)

2. Express the curvature in terms of \( t \).
\[
\mathbf{r}(t) = 2t \mathbf{i} + 3t^3 \mathbf{j}
\]
The Curvature of a Space Curve
A space curve bends in two ways. It bends in the osculating plane (the plane of the unit tangent \( T \) and the principal normal \( N \)) and it bends away from that plane. The first form of bending is measured by the rate at which the unit tangent \( T \) changes direction. The second form of bending is measured by the rate at which the vector \( T \times N \) changes direction. We will concentrate here on the first form of bending, the bending in the osculating plane. The measure of this is called curvature.

If the space curve is given in terms of a parameter \( t \), \( C: \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}, \quad t \in [a, b] \)

Then the curvature is: \( \kappa = \frac{\|d\mathbf{T}/dt\|}{ds/dt} \)

Example: Calculate the curvature of \( \mathbf{r}(t) = \cos t \mathbf{i} + t \mathbf{j} + \sin t \mathbf{k} \)
Components of Acceleration

Given \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \Rightarrow \mathbf{T} = \frac{d\mathbf{r}}{dt} / \|d\mathbf{r}/dt\| = \frac{\mathbf{v}}{ds/dt} \), so: \( \mathbf{v} = \frac{d\mathbf{r}}{dt} \).

This means that \( \mathbf{a} = \mathbf{v}'(t) = \mathbf{T}'(t) \frac{ds}{dt} + \mathbf{T}(t) \frac{d}{dt} \left( \frac{ds}{dt} \right) \) and since \( \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \Rightarrow \mathbf{T}'(t) = \mathbf{N}(t)\|\mathbf{T}'(t)\| \)

So, \( \mathbf{a}(t) = \mathbf{N}(t)\|\mathbf{T}'(t)\| \frac{ds}{dt} + \mathbf{T}(t) \frac{d^2s}{dt^2} \).

The acceleration vector lies in the osculating plane, the plane of \( \mathbf{T} \) and \( \mathbf{N} \). The tangential component of acceleration is given by:

\[ a_T = \mathbf{T} \cdot \mathbf{a} = \frac{\mathbf{v} \cdot \mathbf{a}}{ds/dt} \]

And this depends only on the change of speed of the object; if the speed is constant, the tangential component of acceleration is zero and the acceleration is directed entirely toward the center of curvature of the path.

The normal component of acceleration is

\[ a_N = \mathbf{N} \cdot \mathbf{a} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{ds/dt} \]

And depends both on the speed of the object and the curvature of the path. At a point where the curvature is zero, the normal component of acceleration is zero and the acceleration is directed entirely along the path of motion. If the curvature is not zero, then the normal component of acceleration is a multiple of the square of the speed.

If we take the dot product of \( \mathbf{T} \) with \( \mathbf{a} \), we get

\[ \mathbf{T} \cdot \mathbf{a} = a_T (\mathbf{T} \cdot \mathbf{T}) + a_N (\mathbf{T} \cdot \mathbf{N}) = a_T \]

So, \( a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{ds/dt} \) and \( a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{ds/dt} \).

Which means we can write: \( \kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{(ds/dt)^3} \).
Examples:

1. Interpret $\mathbf{r}(t)$ as the position of a moving object at time $t$. Determine the tangential and normal components of acceleration.

   $\mathbf{r}(t) = \cos(t)\mathbf{i} + t\mathbf{j} + \sin(t)\mathbf{k}$
2. The vector function \( \mathbf{r}(t) = (\cos t + t \sin t) \mathbf{i} + (\sin t - t \cos t) \mathbf{j} + \frac{\sqrt{3}}{2} t^2 \mathbf{k} \) determines a curve \( C \) in space.

(a) Find the unit tangent vector \( T(t) \) and the principal normal vector \( N \).

(b) Find the curvature, \( \kappa \) of \( C \).

(c) Determine the tangential and normal components of acceleration; express the acceleration vector, \( \mathbf{a}(t) \) in terms of \( \mathbf{T} \) and \( \mathbf{N} \).