- All assignments which were originally due before today, are now open until 11:59 p.m. on Thursday 9/14.
LecPop03_1

1. Two lines are parallel if
   a. They don’t intersect
   b. They intersect
   c. Their direction vectors are scalar multiples of each other
   d. The dot product of their direction vectors is 0
   e. None of these

2. Scalar parametric equations for the line that passes through \(P(3,7,2)\) and is parallel to the line: \(r(t) = (2,-1,4) + t(3,4,-6)\) are:
   a. \(\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-4}{-6}\)
   b. \(x = 2 + 3t, y = -1 + 4t, z = 4 - 6t\)
   c. \(\frac{x-3}{3} = \frac{y-7}{4} = \frac{z-2}{-6}\)
   d. \(x = 3 + 3t, y = 7 + 4t, z = 2 - 6t\)
   e. None of these
12.6 Lines

Vector form: \( \mathbf{r}(t) = (x_0 \mathbf{i} + y_0 \mathbf{j} + z_0 \mathbf{k}) + t(d_1 \mathbf{i} + d_2 \mathbf{j} + d_3 \mathbf{k}) \)

\[ \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{d} \]

Useful

Scalar form:
\[ x(t) = x_0 + td_1 \]
\[ y(t) = y_0 + td_2 \]
\[ z(t) = z_0 + td_3 \]

Symmetric form:
\[ \frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3} \]

Rewrite these in one of above forms.
If we have two lines, we can determine if they are parallel, coincident, skew, or intersecting.

Parallel – Parallel but not intersecting

Coincident – Parallel and intersecting

Skew – Neither parallel nor intersecting

Intersect – Not parallel but they do intersect
Once again, two lines \( (l_1: \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{d} \text{ and } l_2: \mathbf{R}(u) = \mathbf{R}_0 + u \mathbf{D}) \) intersect if there exists numbers \( t \) and \( u \) such that \( \mathbf{r}(t) = \mathbf{R}(u) \)

8) Determine whether the lines \( l_1 \) and \( l_2 \) are parallel, coincident, skew, or intersecting. If they intersect, find the point of intersection:

\[
 l_1: x_1(t) = 1 + t, \ y_1(t) = -1 - t, \ z_1(t) = -4 + 2t \\
 l_2: x_2(u) = 1 - u, \ y_2(u) = 1 + 3u, \ z_2(u) = 2u
\]

Are the lines \( l_1 \) and \( l_2 \) parallel?

\( l_1 \neq \alpha l_2 \Rightarrow l_1 \cap \cap l_2 \)

Are the lines \( l_1 \) and \( l_2 \) intersecting?

Yes!

\begin{align*}
\text{1) } & 1+t = 1-u \\
\text{2) } & u = t - 2 \\
\text{3) } & -4+2t = 2u \\
\text{1) } & 1+t = 1-(t-2) = 1-t+2 = 3-t \\
\text{1+} & t = 3-t \Rightarrow 2t = 2 \Rightarrow t = 1 \\
\text{if } t &= 1 \Rightarrow u = 1-2 = -1 \end{align*}

Use (2) to check:

\( -1-t = 1+3u \)

When \( t = 1, u = -1 ? \)  Yes!

\( -1-t = -2, 1+3(-1) = 1-3 = -2 \)

The POI is

\( (x_1(1), y_1(1), z_1(1)) = (x_2(-1), y_2(-1), z_2(-1)) = (2, -2, -2) \)
If two lines are not parallel and do not intersect, then they are skew.

9) Determine whether the lines $l_1$ and $l_2$ are parallel, coincident, skew, or intersecting. If they intersect, find the point of intersection:

$l_1 : x_1(t) = 3 + 2t, y_1(t) = -1 + 4t, z_1(t) = 2 - t$

$l_2 : x_2(u) = 3 + 2u, y_2(u) = 2 + u, z_2(u) = -2 + 2u$

Parallel? $\overrightarrow{z_1} = (2, 1, 2), \overrightarrow{z_2} = (2, 1, 2)$, $\overrightarrow{z_1} \neq \alpha \overrightarrow{z_2} \Rightarrow$ not parallel

No

Intersect? Set

1. $3 + 2t = 3 + 2u$
2. $-1 + 4t = 2 + u$
3. $2 - t = -2 + 2u$

$\Rightarrow t = u$

$\Rightarrow -1 + 4t = 2 + t$
$
\Rightarrow 3t = 3 \Rightarrow t = 1 \Rightarrow u = 1$

Check using 3

$2 - 1 = 1, -2 + 2 \cdot 1 = 0$

Since $1 \neq 0$

We have no intersection

Thus, $l_1$ and $l_2$ are skew
To find the angle between two lines:

\[ \cos \theta = |\mathbf{u}_d \cdot \mathbf{u}_D| \]

\[ \cos \theta = \left| \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| \cdot ||\mathbf{b}||} \right| = \left| \mathbf{u}_d \cdot \mathbf{u}_D \right| \]

Note that the absolute value signs give us the angle \( \theta \in [0, \frac{\pi}{2}] \).
Distance from a point to a line:

\[ d(P, l) = \frac{\|P_0 P_1 \times d\|}{\|d\|} \]

Recall: \[ \|a \times b\| = \|a\| \|b\| \sin \theta \]

\[
\text{dist} = \frac{\|P_0 P_1\| \sin \theta}{\|d\|}
= \frac{\|P_0 P_1\| \|a\| \sin \theta}{\|d\| \|a\|}
= \frac{\|P_0 P_1 \times a\|}{\|a\|}
\]
10) Find the distance from $P_1(2,0,2)$ to the line through $P_0(3,-1,1)$ parallel to $i - 2j - 2k$.

$$d(P,l) = \frac{\|P_0P_1 \times d\|}{\|d\|}$$

Let $\mathbf{d} = (1,-2,-2)$, $\|\mathbf{d}\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$

$P_0P_1 = (2-3,0+1,2-1) = (-1,1,1)$

$P_0P_1 \times d = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 1 & -2 & -2 \end{vmatrix} = \mathbf{i}(-2+2) - \mathbf{j}(2-1) + \mathbf{k}(2-1) = 0\mathbf{i} - \mathbf{j} + \mathbf{k}

= (0,-1,1)$

$$\text{dist} = \frac{\|P_0P_1 \times d\|}{\|d\|} = \frac{\sqrt{0^2 + 1^2 + 1^2}}{3} = \frac{\sqrt{2}}{3}$$
12.7 Planes

Let \( \mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \) be the nonzero vector perpendicular to a plane:

\[
\mathbf{N} \text{ is the Normal Vector}
\]

Point \( P(x_0, y_0, z_0) \) is our known initial point.

Point \( Q(x, y, z) \) is any general point on the plane.

Movement from \( P \) to \( Q \) is of form \( \mathbf{d} = (x-x_0, y-y_0, z-z_0) \).

Vectors \( \mathbf{N} \) and \( \mathbf{d} \) are normal.

\[
\Rightarrow \mathbf{N} \perp \mathbf{d} \Rightarrow 0 = \mathbf{N} \cdot \mathbf{d}
\]

So
\[
0 = (A, B, C) \cdot (x-x_0, y-y_0, z-z_0)
\]

or
\[
0 = A(x-x_0) + B(y-y_0) + C(z-z_0)
\]
The equation of the plane containing \( P(x_0, y_0, z_0) \) with normal \( \mathbf{N} = Ai + Bj + Ck \) is

\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0
\]
Examples:
1) Find an equation for the plane which passes through the point \( P(2, 4, 0) \) and is perpendicular to \( \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \).

\[
A(x-x_0) + B(y-y_0) + C(z-z_0) = 0
\]

Point: \( P(2, 4, 0) \)
Normal: \( \mathbf{N} = \langle 1, -3, 2 \rangle \)

Plane: \[
1(x-2) - 3(y-4) + 2(z-0) = 0
\]
\[
x-2 - 3(y-4) + 2z = 0
\]
or \[
x - 3y + 2z = -10
\]
2) Find an equation for the plane which passes through the point $P(3, -2, 3)$ and is parallel to the plane:

$$4x + 3y + 4z + 6 = 0$$

Is $P$ on this plane?

$$4(3) + 3(-2) + 4(3) + 6 = 24 
eq 0$$

No!

Point: $P(3,-2,3)$

Normal: Parallel planes have parallel normal vectors

So, use $\vec{N} = (4, 3, 4)$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

Plane: $4(x-3) + 3(y+2) + 4(z-3) = 0$

or  $4x + 3y + 4z = 18$
3) Find an equation for the plane which passes through the point $P(1, 3, 4)$ and contains the line:

$$l: x(t) = 3t, \quad y(t) = 4t, \quad z(t) = 2 + 2t$$

**Point:** Use $P(1, 3, 4)$ or any point on $l$

**Normal:** We know two directions of movement in the plane. First $\mathbf{a} = (3, 4, 2)$ and $\overrightarrow{P_0P} = (1, 3, 2)$.

Thus $\overrightarrow{P_0P} \times \mathbf{a}$ is $\perp$ to both.

$$\overrightarrow{P_0P} \times \mathbf{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ 3 & 4 & 2 \end{vmatrix} = \hat{i}(6-8) - \hat{j}(2-6) + \hat{k}(4-9)$$

$$\begin{align*}
&= -2\hat{i} + 4\hat{j} - 5\hat{k} \\
&= (-2, 4, -5)
\end{align*}$$

**Plane:**

$$-2(x-1) + 4(y-3) - 5(z-4) = 0$$
5) Write the equation of the plane in intercept form and find the points where it intersects the coordinate axes.

\[ 4x - 2y + 5z = 20 \]

**Intercept Form:** \[ \frac{x}{5} + \frac{y}{10} + \frac{z}{4} = 1 \]

\[
\frac{4x - 2y + 5z}{20} = \frac{20}{1}
\]

\[
\frac{1}{5}x - \frac{1}{10}y + \frac{1}{4}z = 1
\]

\[
\frac{x}{5} + \frac{y}{10} + \frac{z}{4} = 1
\]

\( (5,0,0), (0,-10,0), (0,0,4) \)
Angle between two planes: \[ \cos \theta = \left| \mathbf{u}_{N_1} \cdot \mathbf{u}_{N_2} \right| = \left| \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{||\mathbf{N}_1|| \cdot ||\mathbf{N}_2||} \right| \]

*Note: Abs value gives \( \theta \in [0, \frac{\pi}{2}] \)*
6) Find the angle between the planes.

\[2x - 2y + 2z + 5 = 0 \text{ and } 2x - y + z - 3 = 0\]

\[
\vec{N}_1 = (2, -2, 2), \quad \vec{N}_2 = (2, -1, 1)
\]

\[
\|\vec{N}_1\| = \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3}
\]

\[
\|\vec{N}_2\| = \sqrt{4 + 1 + 1} = \sqrt{6}
\]

\[
\cos \theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{\|\vec{N}_1\| \|\vec{N}_2\|} = \frac{|4 + 2 + 2|}{2\sqrt{3} \cdot \sqrt{6}} = \frac{8}{2\sqrt{18}} = \frac{4}{3\sqrt{2}} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}
\]

\[
\theta = \arccos \left( \frac{2\sqrt{2}}{3} \right)
\]
Distance from a point to a plane \(d(P, \mathcal{P}) = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}\).

Note that \(\mathcal{P}\) is the plane \(Ax + By + Cz + D = 0\). We must write plane in this form!

7) Find the distance from the point \(P(1, -1, 2)\) to the plane

\[2x - y + 2z + 3 = 0\]

Plane is \(2x - y + 2z + 3 = 0\), \(\vec{N} = (2, -1, 2)\)
\(A = 2, B = -1, C = 2, D = 3\)

\[
\text{Dist} = \frac{|2 \cdot 1 - 1 \cdot (-1) + 2 \cdot 2 + 3|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{|2 + 1 + 4 + 3|}{\sqrt{9}}
\]

\[= \frac{10}{3}\]
8) Find an equation in $x, y, z$ for the plane that passes through the given points.

$P(1, -1, 2), Q(3, -1, 3), R(2, 0, 2)$

**Point:** I like $P(1, -1, 2)$

**Normal:** $\vec{N}$ is $\perp \vec{a}, \vec{b}$

$\vec{a} = \langle 2, 0, 1 \rangle$, $\vec{b} = \langle 1, 1, 0 \rangle$

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(2-0) = -\hat{i} + \hat{j} + 2\hat{k}$

Use $\vec{N} = (-1, 1, 2)$ and $P(1, -1, 2)$

Plane: $-(x-1) + (y+1) + 2(z-2) = 0$

or $-x + y + 2z = 2$
Two planes will be parallel if their normal vectors are scalar multiples of each other. If two planes are not parallel, then they will intersect in a line. The line of intersection will have a direction vector parallel to the cross product of their normal vectors. 

\[ \vec{a} \perp \vec{N}_1, \vec{N}_2 \text{ or } \vec{a} \parallel \vec{N}_1 \times \vec{N}_2 \]

9) Find a set of scalar parametric equations for the line formed by the two intersecting planes.

\[ x + y + z + 1 = 0 \text{ and } 3x - 3y + 3z + 6 = 0 \]

\[ \frac{x - y + z + 2 = 0}{\underline{x - y + z + 2 = 0}} \]

\[ \vec{N}_1 = \langle 1, 1, 1 \rangle, \vec{N}_2 = \langle 1, -1, 1 \rangle, \vec{N}_1 \neq \alpha \vec{N}_2 \]

Find \[ \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(1+1) - \hat{j}(1-1) + \hat{k}(1-1) = 2\hat{i} - 2\hat{k} = \langle 2, 0, -2 \rangle \]

Direction: use \[ \vec{a} = \langle 1, 0, -1 \rangle \]

Point: We need a point \( (x_0, y_0, z_0) \) which is on BOTH planes.

That is \[ x_0 + y_0 + z_0 + 1 = 0 \] \( \rightarrow \) \( (x_0, y_0, z_0) \) on plane 1
\[ x_0 - y_0 + z_0 + 2 = 0 \] \( \rightarrow \) \( (x_0, y_0, z_0) \) on plane 2
\[ x_0 + y_0 + z_0 + 1 = 0 \]
\[ x_0 - y_0 + 2z_0 + 2 = 0 \]

Choose \( x_0 = 0 \)

Then \( y_0 + z_0 = -1 \)
\[
\begin{align*}
-(-y_0 + z_0 &= -2) \\
0 + 2z_0 &= -3 \\
\Rightarrow z_0 &= -\frac{3}{2} \Rightarrow y_0 &= \frac{1}{2}
\end{align*}
\]

Point: \( (0, \frac{1}{2}, -\frac{3}{2}) \)

Direction: \( (1, 0, -1) \)

Line: \( x = 0 + t, y = \frac{1}{2} + 0t, z = -\frac{3}{2} - t \)

or \( x = t, y = \frac{1}{2}, z = -\frac{3}{2} + t \)
LecPop03 - 1

3 E
4 A
5 B