11.3 – The Dot Product

The angle between two vectors is found with this formula: \( \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|a\|\|b\|} \)

3) Given, \( \mathbf{a} = 4\mathbf{i} + 4\mathbf{j}, \quad \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{c} = 2\mathbf{i} + 2\mathbf{k} \)

Find the angle between \( \mathbf{a} \) and \( \mathbf{c} \)
Projection of \( \mathbf{a} \) on \( \mathbf{b} \) \( \text{proj}_b \mathbf{a} = (\mathbf{a} \cdot \mathbf{u}_b) \mathbf{u}_b \) where \( \mathbf{a} \cdot \mathbf{u}_b = \text{comp}_b \mathbf{a} \)

4) Given \( \mathbf{a} = 4\mathbf{i} + 3\mathbf{j} \), \( \mathbf{b} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \)

Find the component of \( \mathbf{a} \) in the \( \mathbf{b} \) direction.

Find the projection of \( \mathbf{a} \) in the \( \mathbf{b} \) direction.
The angles $\alpha, \beta, \gamma$ that a vector makes with unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called direction angles of $\mathbf{a}$. A unit vector with these direction angles is: $\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$

5) Find a unit vector with direction angles: $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{2}$

6) Find the direction angles of $\mathbf{a} = \mathbf{i} - \sqrt{3} \mathbf{k}$
7) Find all the numbers \( x \) for which the angle between \( \mathbf{c} = xi + j + k \) and \( \mathbf{d} = i + xj + k \) is \( \frac{\pi}{3} \).

8) Find all numbers \( x \) such that

\[
2i + 4j + 2xk \perp 6i + 3j - 4xk
\]
11.4 – The Cross Product

What operations can we use on vectors?

If vectors $\mathbf{a}$ and $\mathbf{b}$ are NOT parallel, they form two adjacent sides of a parallelogram:

$a \times b$ is the vector perpendicular to this plane.
So, if $\mathbf{a} \parallel \mathbf{b}$ then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

The area of this parallelogram is $A = \| \mathbf{a} \times \mathbf{b} \|$

How would we find the area of the triangle that has $\mathbf{a}$ and $\mathbf{b}$ as sides?
So, how do we find \( \mathbf{a} \times \mathbf{b} \)?

First we need to understand determinants:

\[
\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_2 b_2 - a_2 b_1
\]

for \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), \( \mathbf{a} \times \mathbf{b} = \)

Example

1) Calculate \((3 \mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 2 \mathbf{j})\)
Example
2) Calculate $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot [(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})]

3) Find two unit vectors perpendicular to $\mathbf{a} = (1,2,-1)$ and $\mathbf{b} = (1,0,2)$
4) Find the area of triangle $PQR$.

$$P(2, 0, -3), \quad Q(3, -1, 0), \quad R(2, 3, -2)$$

5) Find the volume of the parallelepiped with the given edges.

$$i - 3j + k, \quad 3j - k, \quad i + j - 3k$$

How can we tell if 3 vectors are coplanar?
11.5 – Lines

Vector form: \( \mathbf{r}(t) = (x_0 \mathbf{i} + y_0 \mathbf{j} + z_0 \mathbf{k}) + t(d_1 \mathbf{i} + d_2 \mathbf{j} + d_3 \mathbf{k}) \)

\[
\begin{align*}
x(t) &= x_0 + td_1 \\
y(t) &= y_0 + td_2 \\
z(t) &= z_0 + td_3
\end{align*}
\]

Scalar form: \( y(t) = y_0 + td_2 \)

\( z(t) = z_0 + td_3 \)

Symmetric form: \( \frac{x-x_0}{d_1} = \frac{y-y_0}{d_2} = \frac{z-z_0}{d_3} \)

Examples:
1) Find a vector parameterization for the line that passes through \( P(3, 2, 3) \) and is parallel to the line \( \mathbf{r}(t) = (\mathbf{i} + \mathbf{j} - \mathbf{k}) + t(2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}) \).

2) Give the answer to #1 in scalar form

3) Give the answer to #1 in symmetric form
4) Give the symmetric form for the line that passes through \( P(2, 2, 1) \) and \( Q(2, -2, -3) \).

5) Find a set of scalar parametric equations for the line that passes through \( P(1, 2, -3) \) and is perpendicular to the \( xy \)-plane.

Parallel lines have direction vectors that are scalar multiples of each other.

6) Give a vector parameterization for the line that passes through \( P(1, 2, -2) \) and is parallel to the line: 
\[ 3(x - 1) = 2(y - 2) = 6(z + 2) \]
If we have two lines, we can determine if they are parallel, coincident, skew, or intersecting.

Parallel –

Coincident –

Skew –

Intersect –

7) Determine whether the lines $l_1$ and $l_2$ are parallel, coincident, skew, or intersecting. If they intersect, find the point of intersection:

$l_1 : \mathbf{r}(t) = (-i + 2j + k) + t(i - 3j + 2k)$

$l_2 : \mathbf{R}(u) = (2i - j) + u(-2i + 6j - 4k)$
Once again, two lines \( l_1: \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{d} \) and \( l_2: \mathbf{R}(u) = \mathbf{R}_0 + u \mathbf{D} \) intersect if there exists numbers \( t \) and \( u \) such that \( \mathbf{r}(t) = \mathbf{R}(u) \)

8) Determine whether the lines \( l_1 \) and \( l_2 \) are parallel, coincident, skew, or intersecting. If they intersect, find the point of intersection:

\[
\begin{align*}
\mathbf{l}_1 & : x_1(t) = 1 + t, \quad y_1(t) = -1 - t, \quad z_1(t) = -4 + 2t \\
\mathbf{l}_2 & : x_2(u) = 1 - u, \quad y_2(u) = 1 + 3u, \quad z_2(u) = 2u
\end{align*}
\]
If two lines are not parallel and do not intersect, then they are skew.

9) Determine whether the lines $l_1$ and $l_2$ are parallel, coincident, skew, or intersecting. If they intersect, find the point of intersection:

\[
l_1 : x_1(t) = 3 + 2t, y_1(t) = -1 + 4t, z_1(t) = 2 - t
\]

\[
l_2 : x_2(u) = 3 + 2u, y_2(u) = 2 + u, z_2(u) = -2 + 2u
\]
To find the angle between two lines:
\[ \cos \theta = |\mathbf{u}_d \cdot \mathbf{u}_D| \]

Distance from a point to a line:

\[
d(P,l) = \frac{||\overrightarrow{P_0P_1} \times \mathbf{d}||}{||\mathbf{d}||}
\]

10) Find the distance from \(P(2, 0, 2)\) to the line through \(P_0(3, -1, 1)\) parallel to \(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\).
11.6 – Planes

Let \( \mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \) be the nonzero vector perpendicular to a plane:

\[
\begin{align*}
N &= Ai + Bj + Ck \\
\end{align*}
\]

The equation of the plane containing \( P(x_0, y_0, z_0) \) with normal \( \mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \) is

\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0
\]

Examples:
1) Find an equation for the plane which passes through the point \( P(2, 4, 0) \) and is perpendicular to \( \mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \).
2) Find an equation for the plane which passes through the point \( P(3, -2, 3) \) and is parallel to the plane:

\[ 4x + 3y + 4z + 6 = 0 \]

3) Find an equation for the plane which passes through the point \( P(1, 3, 4) \) and contains the line:

\[ l: \{x(t) = 3t, y(t) = 4t, z(t) = 2 + 2t\} \]
4) Find the unit normal vectors to the plane: \[ 3x - 4y + 2z - 2 = 0 \]

5) Write the equation of the plane in intercept form and find the points where it intersects the coordinate axes. \[ 4x - 2y + 5z = 20 \]
Angle between two planes: \( \cos \theta = |\mathbf{u}_n \cdot \mathbf{u}_{n_2}| \)

6) Find the angle between the planes. \( 2x - 2y + 2z + 5 = 0 \) and \( 2x - y + z - 3 = 0 \)

Distance from a point to a plane \( d(P, \phi) = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \)

7) Find the distance from the point \( P(1, -1, 2) \) to the plane \( 2x - y + 2z + 3 = 0 \)
8) Find an equation in $x, \, y, \, z$ for the plane that passes through the given points.

$P(1, \, -1, \, 2), \, Q(3, \, -1, \, 3), \, R(2, \, 0, \, 2)$
Two planes will be parallel if their normal vectors are scalar multiples of each other. If two planes are not parallel, then they will intersect in a line. The line of intersection will have a direction vector equal to the cross product of their norms.

9) Find a set of scalar parametric equations for the line formed by the two intersecting planes.

\[ x + y + z + 1 = 0 \quad \text{and} \quad 3x - 3y + 3z + 6 = 0 \]