12.6 Lines

The vector equation \( \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{d} \) \((t \text{ is a real number})\) parameterizes the line \( l \). The tip of \( \mathbf{r}_0 \) is the point \((x_0, y_0, z_0)\) and \( \mathbf{d} \) (the direction vector) is \((d_1, d_2, d_3)\) so we can write \( l \) as:

\[
\lim_{x \to \infty} \mathbf{r}(t) = (x_0 + t \mathbf{d}_1) \mathbf{i} + (y_0 + t \mathbf{d}_2) \mathbf{j} + (z_0 + t \mathbf{d}_3) \mathbf{k}
\]

**Example:**
Find a vector parameterization for the line that passes through the point \( P(1, -2, 0) \) and is parallel to the line \( \mathbf{r}(t) = (\mathbf{i} - \mathbf{j} + 2 \mathbf{k}) + t (\mathbf{i} + \mathbf{k}) \)

\[
\mathbf{d} = \mathbf{i} + \mathbf{k} \text{ or } (1, 0, 1)
\]

\[
\mathbf{r}_p(t) = (1, -2, 0) + t (1, 0, 1)
\]

or

\[
\mathbf{r}_q(t) = (\mathbf{i} - 2 \mathbf{j}) + t (\mathbf{i} + \mathbf{k})
\]

or

\[
\mathbf{r}_p(t) = (1 + t) \mathbf{i} + (-2) \mathbf{j} + t \mathbf{k}
\]
Lines can also be parameterized by a set of scalar equations $x(t)$, $y(t)$ and $z(t)$:

$$\begin{align*}
x(t) &= x_0 + td_1 \\
y(t) &= y_0 + td_2 \\
z(t) &= z_0 + td_3
\end{align*}$$

Symmetric Form:

$$\frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3} \quad \text{if and only if “} \mathbf{d} \text{” are 0, omit a writes } x = x_0, y = y_0 \text{ or } z = z_0$$

Example:

Give scalar equations for the line through the point $P(1, -2, 0)$ with direction $\mathbf{d} = (5, 0, -1)$

$$(x(t) = 1 + 5t, y(t) = -2 + 0t = -2, z(t) = 0 - t = -t)$$

For the same line, give the symmetric form.

$$\frac{x - 1}{5} = \frac{z - 0}{-1} \quad \text{and} \quad y = -2$$
Determining if lines intersect, are skew, are parallel or coincide:

Two lines ($l_1: \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{d}$ and $l_2: \mathbf{R}(u) = \mathbf{R}_0 + u \mathbf{D}$) intersect if there exists numbers $t$ and $u$ such that $\mathbf{r}(t) = \mathbf{R}(u)$.

What would make two lines parallel? $\mathbf{d} = \alpha \mathbf{D}$, no points in common

Coincide? $\mathbf{d} = \alpha \mathbf{D}$ & have pts in common

Skew? not parallel & don’t intersect

Intersect? pt in common

Example:

$l_1: \mathbf{r}(t) = (3i + j + 5k) + t(i - j + 2k)$

$l_2: \mathbf{R}(u) = (i + 4j + 2k) + u(j + k)$

$\mathbf{d} = (1, -1, 2)$ \text{ not scalar multiples}$\Rightarrow$ not parallel

$\mathbf{D} = (0, 1, 1)$

do they intersect? re write in scalar form

\[
\begin{align*}
l_1: \quad & x(t) = 3 + t, \quad y(t) = 1 - t, \quad z(t) = 5 + 2t \\
l_2: \quad & x(u) = 1 + 0u, \quad y(u) = 4 + u, \quad z(u) = 2 + u
\end{align*}
\]

Point of intersection = $(1, 3, 1)$

$3 + t = 1 + 0u$ \use to solve for $t + u$

$1 - t = 4 + u$

$5 + 2t = 2 + u$ \use to check our answer

$3 + t = 1$ \quad $1 - (-2) = 4 + u$

\[
\begin{align*}
\frac{3}{2} = 4 + u \\
\frac{-1}{1} = u
\end{align*}
\]

$\Rightarrow$ intersect

\[
\begin{align*}
t = -2 \\
1 - (-2) = 4 + u \quad \check{5 + 2(-2) = 2 + 1}
\end{align*}
\]
To find the angle between two lines:

$$\cos \theta = \left| u_d \cdot u_p \right|$$

$$\cos \theta = \frac{\left| \mathbf{a} \cdot \mathbf{b} \right|}{\left| \mathbf{a} \right| \left| \mathbf{b} \right|}$$

Line segments – How would we restrict \( t \) so that \( \mathbf{r}(t) \) only traces out a line segment?

Restrict \( t \) so that the equations

\[
\begin{align*}
x(t) &= 7 - 5t, \\
y(t) &= -3 + 2t, \\
z(t) &= 4 - t
\end{align*}
\]

parametrize the line segment that begins at \((12, -5, 5)\) and ends at \((-3, 1, 2)\).

Start:  
\( 7 - 5t = 12 \), \( -3 + 2t = -5 \), \( 4 - t = 5 \)

\[t = -1\]

End:  
\( 7 - 5t = -3 \), \( -3 + 2t = 1 \), \( 4 - t = 2 \)

\[t = 2\]

\[-1 \leq t \leq 2\]
Distance from a point to a line:

\[ d(P, l) = \frac{\|P_0P_1 \times \vec{d}\|}{\|\vec{d}\|} \]

Example:

**P(1,2,3)** \( l: r(t) = \vec{i} + 2\vec{k} + t(\vec{i} - 2\vec{j} + 3\vec{k}) \)

\[ \begin{align*}
P_0 &= (1, 0, 2) \\
P_1 &= (1, 2, 3) \\
\vec{d} &= (1, -2, 3)
\end{align*} \]

\[ \vec{P_0P_1} \times \vec{d} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
0 & 2 & 1 \\
1 & -2 & 3
\end{vmatrix} = \begin{vmatrix}
2 & 1 & 1 \\
0 & 3 & 1 \\
1 & 0 & 2
\end{vmatrix} \vec{i} + \begin{vmatrix}
2 & 1 & 1 \\
0 & 3 & 1 \\
1 & 0 & 2
\end{vmatrix} \vec{j} + \begin{vmatrix}
2 & 1 & 1 \\
0 & 3 & 1 \\
1 & 0 & 2
\end{vmatrix} \vec{k} = 8\vec{i} + \vec{j} - 2\vec{k} \]

\[ d(P, l) = \frac{\|8\vec{i} + \vec{j} - 2\vec{k}\|}{\|\vec{i} - 2\vec{j} + 3\vec{k}\|} = \frac{\sqrt{69}}{\sqrt{14}} \]