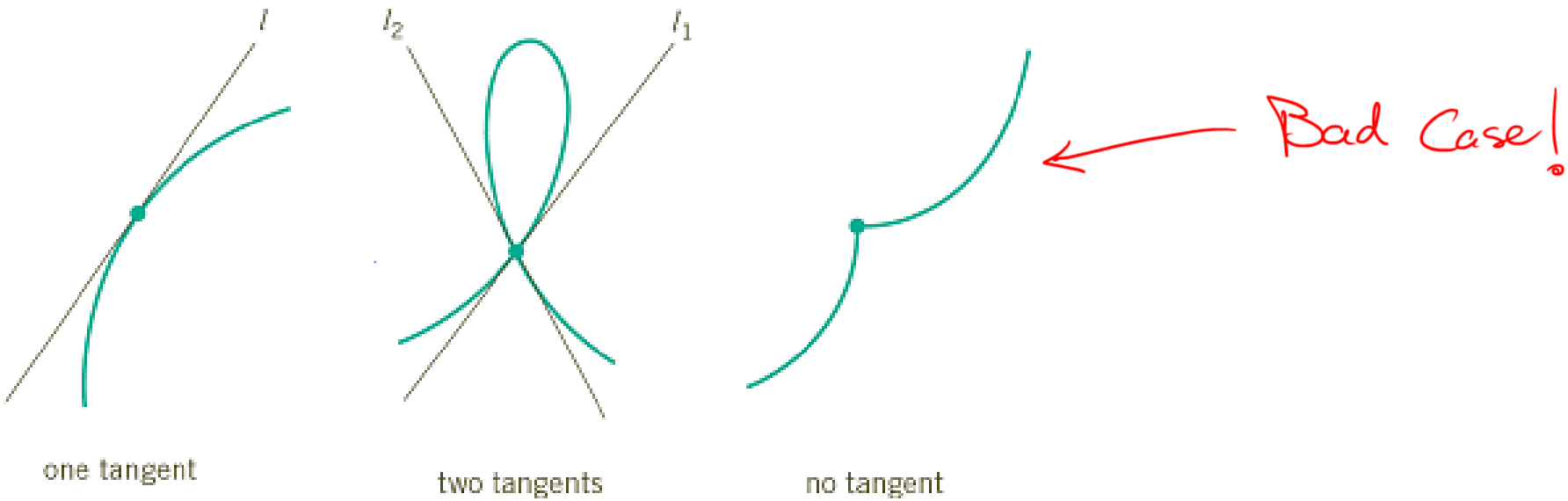
Section 10.4: Tangents to Curves Given Parametrically

Let C be a curve parametrized by the functions

$$x = x(t), \qquad y = y(t)$$

where x and y are defined on some interval I. Since a curve can intersect itself, at any given point C can have

(i) one tangent, (ii) two or more tangents, or (iii) no tangent at all.



To make sure that at least one tangent line exists at each point of C, we will assume that

$$[x'(t)]^2 + [y'(t)]^2 \neq 0.$$

This says there can be no t so that $x'(t) = 0$ and $y'(t) = 0$

Slope of tangent line for parametric curves:

$$m = \frac{y'(t)}{x'(t)}$$

$$m = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

So, equation of tangent line to parametric curve at $t = t_0$ is

$$\underbrace{y - y(t_0) = \frac{y'(t_0)}{x'(t_0)}\left(x - x(t_0)\right)}_{\text{Parametric Form of Point-Slope Formula}}$$

Point-slope formula: $y - y_0 = m(x - x_0)$

here $(x_0, y_0) = (x(t_0); y(t_0))$

Slope is $m = \frac{y'(t_0)}{x'(t_0)}$

Sometime we write:

$$x'(t_0)\left(y - y(t_0)\right) = y'(t_0)\left(x - x(t_0)\right)$$

Examples

2. Find the equation of the tangent line at t = 1 to the curve given by the parametric equations $x(t) = 3t^2 - 4t + 2, \quad y(t) = t^3 - 4t.$

Point: $(x(1), y(1)) = (1, -3)$

Slope: $x'(t) = 6t - 4, \quad y'(t) = 3t^2 - 4$

$x'(1) = 6 - 4 = 2, \quad y'(1) = 3 - 4 = -1$

$m = \frac{y'(1)}{x'(1)} = \frac{-1}{2}$

$$\boxed{\text{Line: } y + 3 = -\tfrac{1}{2}(x - 1)}$$

3. Find an equation in x and y for the line tangent to the polar curve.

$$r = 4\cos(2\theta) \quad \theta = \frac{\pi}{2}$$

$x = r\cos\theta = 4\cos(2\theta)\cdot\cos(\theta)$, $\quad x(\frac{\pi}{2}) = 4\cdot\cos(\pi)\cdot\cos(\pi/2) = 0$

$y = r\sin\theta = 4\cos(2\theta)\sin(\theta)$, $\quad y(\frac{\pi}{2}) = 4\cos(\pi)\cdot\sin(\pi/2) = -4$

Point: $(0, -4)$

Slope: $x'(\theta) = -8\sin(2\theta)\cos\theta - 4\cos(2\theta)\sin(\theta)$

$y'(\theta) = -8\sin(2\theta)\sin(\theta) + 4\cos(2\theta)\cos(\theta)$

$x'(\frac{\pi}{2}) = 0 - 4(-1)\cdot(1) = 4$

$y'(\frac{\pi}{2}) = 0 + 0 = 0$

$m = \frac{y'(\frac{\pi}{2})}{x'(\frac{\pi}{2})} = 0$

Line: $y + 4 = 0(x - 0) = 0 \Rightarrow \boxed{y = -4}$

4. Find all points of horizontal and vertical tangency, given

$x = t^2, \quad y = t^3 - 3t$

$x'(t) = 2t, \quad y'(t) = 3t^2 - 3 = 3(t-1)(t+1)$

Horizontal	Vertical
$y'(t) = 0$ and $x'(t) \neq 0$	$x'(t) = 0$ and $y'(t) \neq 0$
$y'(t) = 0 \Rightarrow t = 1$ or $t = -1$	$x'(t) = 0 \Rightarrow t = 0$
Points: $(x(1), y(1)) = (1, -2)$ $(x(-1), y(-1)) = (1, 2)$	Point: $(x(0), y(0)) = (0, 0)$