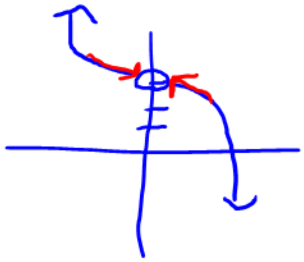
Math 1314 ONLINE
Week 4 Notes

Lesson 4 – Limits (continued)

Last week we looked at limits generally, and at finding limits using substitution.



$\lim_{x \to 0} f(x) = \underline{\underline{3}}$

what is the y-value approaching?

$\lim_{x \to 2} (2x^2 - 7x + 1) = 2(2)^2 - 7(2) + 1$
$= 8 - 14 + 1 = -5$

$\lim_{x \to 3} \frac{2+x}{x-3} = \frac{2+3}{3-3} = \frac{5}{0}$

limit dne

Indeterminate Forms

What do you do when substitution gives you the value $\frac{0}{0}$?

This is called an indeterminate form. It means that we are not done with the problem! We must try another method for evaluating the limit!!

Once we determine that a problem is in indeterminate form, we can use GGB.

Use substitution to see what we have.

What is the command?

Limit [function, value]

Example 8: Determine if the function given is indeterminate. If it is, use GGB to evaluate the limit.

$$\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 - 4} = 1.75$$

$\frac{2^2 + 3(2) - 10}{2^2 - 4} = \frac{4 + 6 - 10}{4 - 4} = \frac{0}{0}$

indeterminate

$\lim_{x \to 2} \frac{(x+5)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$

$\lim_{x \to 2} \frac{x+5}{x+2} = \frac{2+5}{2+2} = \frac{7}{4}$
$= 1.75$

Limit [f, 2]

Example 9: Determine if the function given is indeterminate. If it is, use GGB to evaluate the limit.

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$$

Is it indeterminate? yes

$$\frac{3^2 - 5(3) + 6}{3 - 3} = \frac{9 - 15 + 6}{0} = \frac{0}{0}$$

Limit [f, 3]

$$\boxed{\lim_{x \to 3} f(x) = 1}$$

Try it!! Determine if the function given is indeterminate. If it is, use GGB to evaluate the limit.

$$\lim_{x \to 3} \frac{x^3 - 27}{x - 3}$$

So far we have looked at problems where the target number is a specific real number. Sometimes we are interested in finding out what happens to our function as *x* increases (or decreases) without bound.

$$\lim_{x \to \infty} f(x) =$$

Limits at Infinity

Example 10: Consider the function $f(x) = \dfrac{3x^2}{x^2 + 5}$. What happens to *f*(*x*) as we let the value of *x* get larger and larger?

x	10	50	100	1000	10,000	100,000	1,000,000	10,000,000
$f(x)$								

as we let $x \to \infty$

we see that $y \to 3$

$$\boxed{\lim_{x \to \infty} \frac{3x^2}{x^2 + 5} = 3}$$

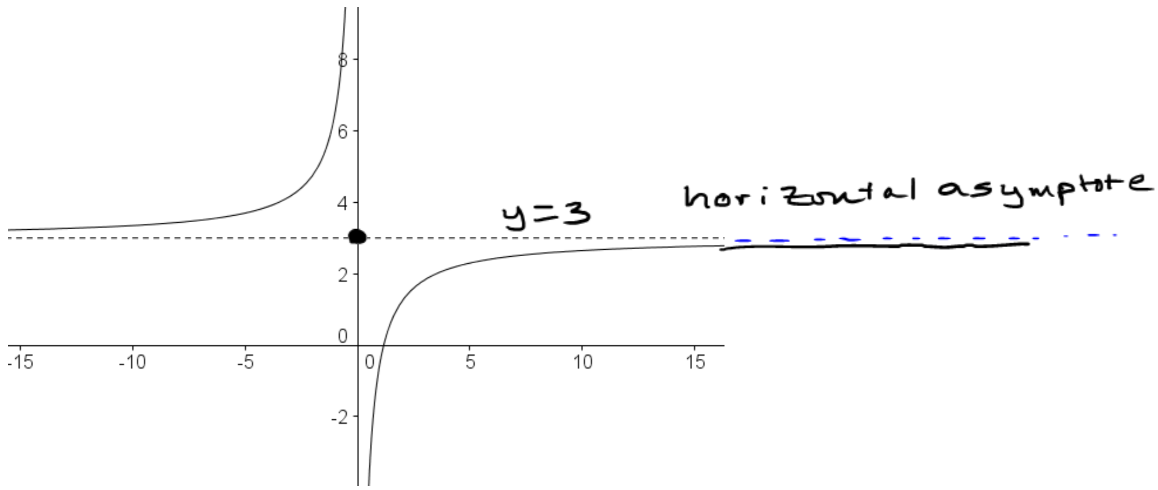
A	B
10	2.857142857142...
50	2.994011976047...
100	2.998500749625...
1000	2.999985000075
10000	2.999999850000...
100000	2.9999999985
1000000	2.999999999985
10000000	2.999999999999...

We say that a function $f(x)$ has the limit L as x increases without bound (or as x approaches infinity), written $\lim_{x \to \infty} f(x) = L$, if $f(x)$ can be made arbitrarily close to L by taking x large enough.

We say that a function $f(x)$ has the limit L as x decreases without bound (or as x approaches negative infinity), written $\lim_{x \to -\infty} f(x) = L$, if $f(x)$ can be made arbitrarily close to L by taking x to be negative and sufficiently large in absolute value.

We can also find a limit at infinity by looking at the graph of a function:

Example 11: Evaluate: $\lim_{x\to\infty}\frac{6x-7}{2x}$ $= 3$



We can also find limits at infinity algebraically:

Example 12: Evaluate: $\lim_{x\to\infty}\left(5x^3-8x-3\right)$

∞

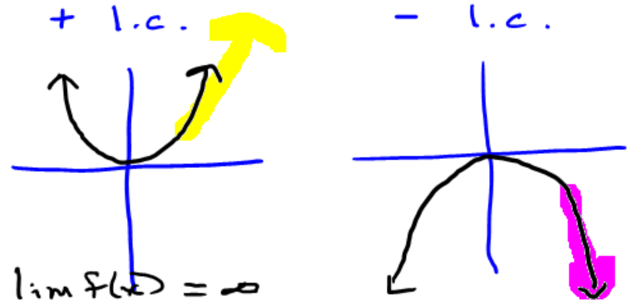
(limit dne and graph of f(x) $\to \infty$ as $x \to \infty$)

look at end behavior

4 possibilities

even degree

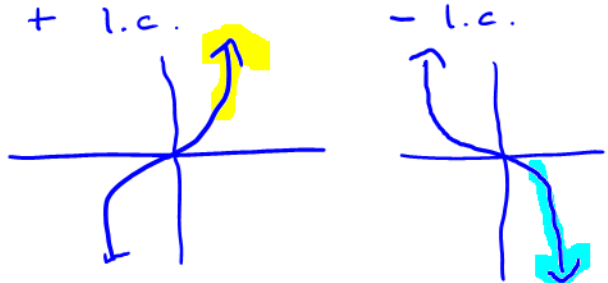
+ l.c.   − l.c.



$\lim_{x\to\infty} f(x) = \infty$ $\lim_{x\to\infty} f(x) = -\infty$

odd degree

+ l.c.   − l.c.



$\lim_{x\to\infty} f(x) = \infty$ $\lim_{x\to\infty} f(x) = -\infty$

Limits at infinity problems often involve rational expressions (fractions). The technique we can use to evaluate limits at infinity is to divide every term in the numerator and the denominator of the rational expression by x^n, where n is the highest power of x present in the denominator of the expression.

Then we can apply this theorem:

Theorem: Suppose $n > 0$. Then $\lim_{x\to\infty} \frac{1}{x^n} = 0$ and $\lim_{x\to-\infty} \frac{1}{x^n} = 0$, provided $\frac{1}{x^n}$ is defined.

After applying this limit, we can determine what the answer should be.

Example 13: Evaluate: $\lim_{x\to\infty} \frac{5x^2+2x+6}{2x^2-7x+1} = \boxed{\frac{5}{2}}$

BOBO
BOTN
EATS DC

Often students prefer to just learn some rules for finding limits at infinity.

The highest power of the variable in a polynomial is called the **degree** of the polynomial. We can compare the degree of the numerator with the degree of the denominator and come up with some generalizations.

If the degree of the numerator is smaller than the degree of the denominator, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

If the degree of the numerator is the same as the degree of the denominator, then you can find $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ by making a fraction from the leading coefficients of the numerator and denominator and then reducing to lowest terms.

If the degree of the numerator is larger than the degree of the denominator, then it's best to work out the problem by dividing each term by the highest power of x in the denominator and simplifying. You can then decide if the function approaches ∞ or $-\infty$, depending on the relative powers and the coefficients.

The notation, $\lim_{x\to\infty} f(x) = \infty$ indicates that, as the value of x increases, the value of the function increases without bound. This limit does not exist, but the ∞ notation is more descriptive, so we will use it.

→ BOBO → 0

→ BOTN → no H.A.

→ EATS DC → a/b

more work

Example 14: Evaluate: $\lim_{x\to-\infty} \dfrac{3x^3+2x-4}{x^2+2x+5}$

divide all terms by the highest power of x in denom.

$$\lim_{x\to\infty} \frac{\frac{3x^3}{x^2} + \frac{x}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{5}{x^2}} = \lim_{x\to\infty} \frac{3x + \underbrace{\frac{1}{x}}_{\to 0} - \underbrace{\frac{4}{x^2}}_{\to 0}}{1 + \underbrace{\frac{2}{x}}_{\to 0} + \underbrace{\frac{5}{x^2}}_{\to 0}}$$

use theorem

$$\lim_{x\to\infty} (3x) = \infty$$

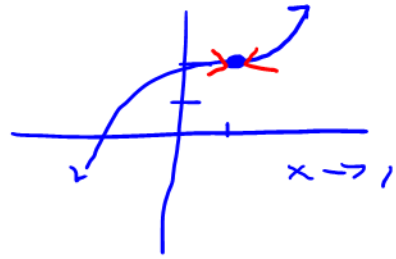
Example 15: Evaluate: $\lim_{x\to\infty} \dfrac{9x^2-x+1}{5x^2+2x+5} = \boxed{\dfrac{9}{5}}$ EATS DC

Example 16: Evaluate : $\lim_{x\to\infty} \dfrac{x-7}{x^2+9x-3} = \boxed{0}$ BOBO

Example 17: Evaluate: $\lim_{x\to-\infty} \dfrac{3x^4-4x+5}{2x\ \ -7x^2}$ BOT N

$$\lim_{x\to-\infty} \frac{\frac{3x^4}{x^2} - \frac{4x}{x^2} + \frac{5}{x^2}}{\frac{2x}{x^2} - \frac{7x^2}{x^2}} = \lim_{x\to-\infty} \frac{3x^2 - \underbrace{\frac{4}{x}}_{\to 0} + \underbrace{\frac{5}{x^2}}_{\to 0}}{\underbrace{\frac{2}{x}}_{\to 0} - 7}$$

$$\lim_{x\to-\infty} \left(-\frac{3}{7}x^2\right) = -\infty$$

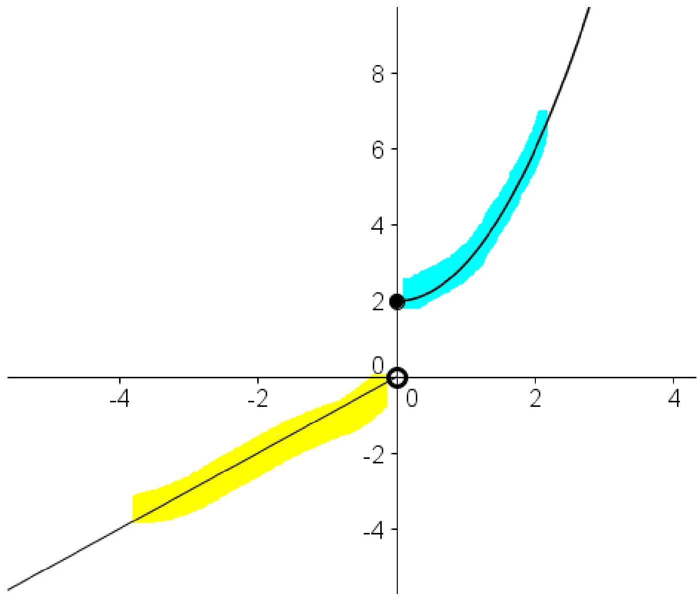


Lesson 5: One-Sided Limits and Continuity

One Sided Limits

Sometimes we are only interested in the behavior of a function when we look from one side and not from the other.

Example 1: Consider the function that is graphed below. Find $\lim_{x \to 0} f(x)$. dne



Now suppose we are *only* interested in looking at the values of x that are bigger than 0. In this case, we are looking at a **one-sided limit**.

We write $\lim_{x \to 0^+} f(x)$. This is called a right-hand limit, because we are looking at values on the right side of the target number.

In this case, $\boxed{\lim_{x \to 0^+} f(x) = 2}$

If we are interested in looking only at the values of x that are smaller than 0, then we would be finding the left-hand limit. The values of x that are smaller than 0 are to the left of 0 on the number line, hence the name. We write $\lim_{x \to 0^-} f(x)$.

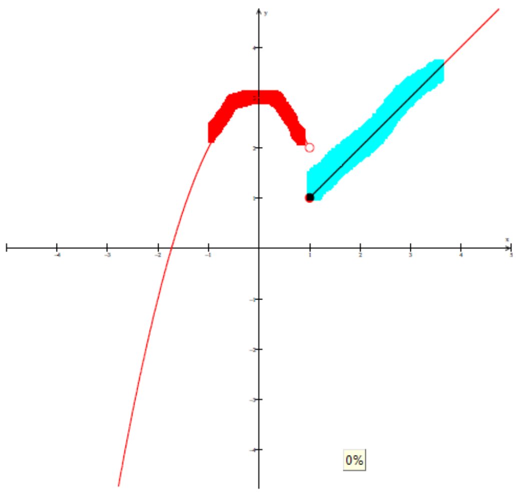
In this case, $\boxed{\lim_{x \to 0^-} f(x) = 0}$

$\boxed{\lim_{x \to 0} f(x) \text{ dne}}$

Our definition of a limit is consistent with this information. We say that $\lim_{x \to a} f(x) = L$ if and only if the function approaches the same value, L, from both the left side and the right side of the target number. This idea is formalized in this theorem:

Theorem: Let f be a function that is defined for all values of x close to the target number a, except perhaps at a itself. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Example 2: Consider this graph:



$\lim_{x \to 1^-} f(x) = 2$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1} f(x)$ dne

Find $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to 1} f(x)$, if it exists.

We can also find one-sided limits from piecewise defined functions.

Example 3: Suppose $f(x) = \begin{cases} x+4, & x > 2 \\ x^2 - 1, & x \le 2 \end{cases}$. Find $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2} f(x)$ (if it exists).

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 1) = 2^2 - 1 = 4 - 1 = 3$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x+4) = 2 + 4 = 6$

$\lim_{x \to 2} f(x)$ dne

Example 4: Suppose $f(x) = \begin{cases} x^2+3, & x<-1 \\ x^3+5, & x \geq -1 \end{cases}$.

Find $\lim_{x\to -1^-} f(x)$, $\lim_{x\to -1^+} f(x)$ and $\lim_{x\to -1} f(x)$ (if it exists).

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+3) = (-1)^2+3 = 1+3=4$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x^3+5) = (-1)^3+5 = -1+5=4$

$\lim_{x\to -1} f(x) = 4$

Example 5:

Suppose $f(x) = \begin{cases} x^2-x+2, & x<1 \\ x+1, & 1 \leq x < 2 \\ x^3-4, & x \geq 2 \end{cases}$.

A. Find $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1} f(x)$ (if it exists).

B. Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2} f(x)$ (if it exists).

A. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2-x+2) = 1^2-1+2 = 2$

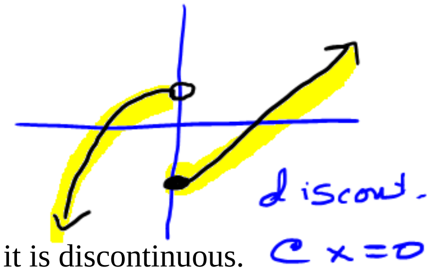
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = 1+1=2$

$\lim_{x\to 1} f(x) = 2$

B. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+1) = 2+1=3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^3-4) = 2^3-4 = 8-4 = 4$

$\lim_{x\to 2} f(x)$ dne



Continuity

We will be interested in finding where a function is continuous and where it is discontinuous. We'll look at continuity over the entire domain of the function, over a given interval and at a specific point.

Continuity at a Point:

Here's the general idea of continuity at a point: a function is continuous at a point if its graph has no gaps, holes, breaks or jumps at that point. Stated a bit more formally,

A function *f* is said to be continuous at the point $x = a$ if the following three conditions are met:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

You'll need to check each of these three conditions to determine if a function is continuous at a specific point.

Example 7: Determine if $f(x) = \begin{cases} 5x - 1, & x \geq 1 \\ x^2 + 3, & x < 1 \end{cases}$ is continuous at $x = 1$.

① $f(1)$ exists? $f(1) = 5(1) - 1 = 4$

yes, it's $\underline{\underline{4}}$

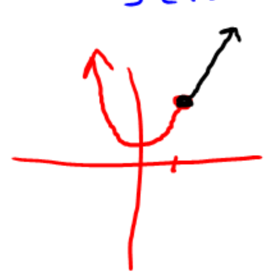
② $\lim_{x \to 1} f(x)$ exists

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 3) = 1^2 + 3 = 4$

yes, it's $\underline{\underline{4}}$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5x - 1) = 5(1) - 1 = 4$

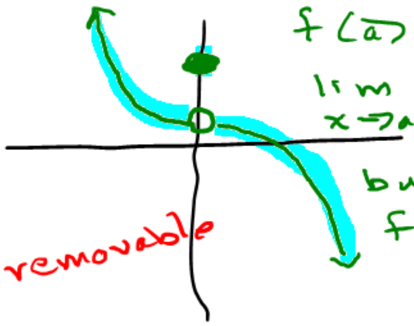
③ $f(1) = \lim_{x \to 1} f(x) = 4$



$\boxed{\text{Yes, } f \text{ is cont @ } x = 1}$

If a function is not continuous at a point, then we say it is discontinuous at that point.

We find points of discontinuity by examining the function that we are given using the three-part continuity test. A discontinuity can be a removable discontinuity, a jump discontinuity or an infinite discontinuity.



$f(a)$ exists

$\lim_{x \to a} f(x)$ exists

but

$f(a) \neq \lim_{x \to a} f(x)$

$f(a)$ exists

$\lim_{x \to a} f(x)$ dne

$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

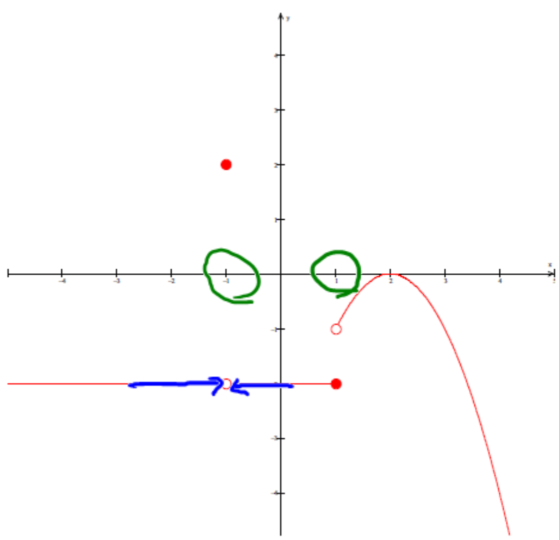
$f(a)$ exists

$\lim_{x \to a} f(x)$ dne

$\lim_{x \to a^-} f(x)$ dne

$\lim_{x \to a^+} f(x)$ dne

Example 8: Find any points of discontinuity. State why the function is discontinuous at each point of discontinuity and state the type of discontinuity.



$x = \pm 1$

$x = -1$ $\quad f(-1) = 2$

$\lim_{x \to -1} f(x) = -2$

$f(-1) \neq \lim_{x \to -1} f(x)$

removable disc

$x = 1$ $\quad f(1) = -2$

$\lim_{x \to 1^-} f(x) = -2$

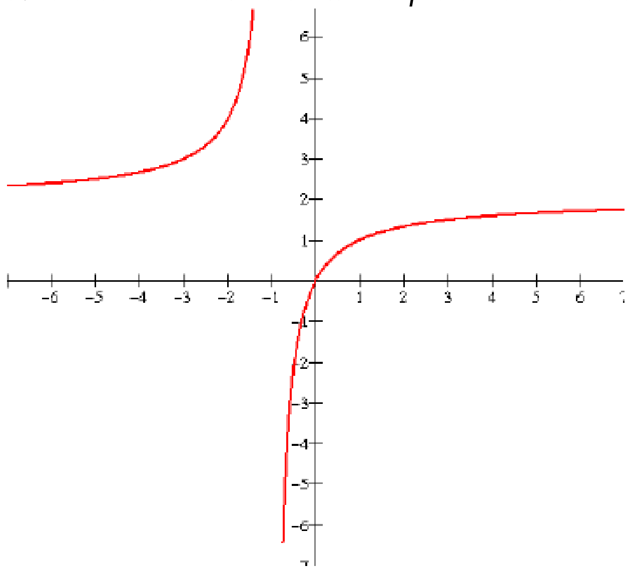
$\lim_{x \to 1^+} f(x) = -1$

$\boxed{\lim_{x \to 1} f(x) \text{ dne}}$

$\boxed{\text{jump discont.}}$

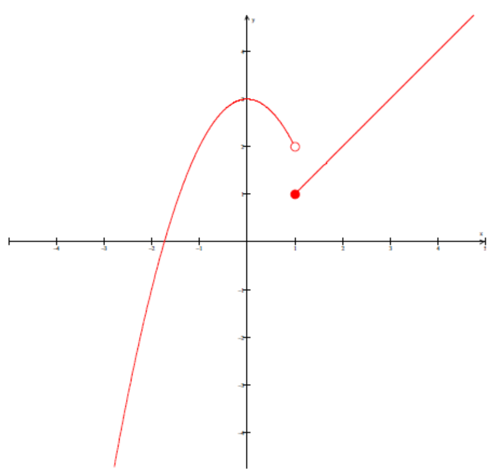
Sometimes, all we want to do is to state the intervals on which a function is continuous.

Example 9: State the intervals on which *f* is continuous using interval notation:



What type of discontinuity is present in example 9?

Example 10: State where *f* is continuous using interval notation:



What type of discontinuity is present in example 10?

Example 11: State where f is continuous using interval notation: $f(x) = \dfrac{5x-10}{x^2-25}$

What type of discontinuity is present in example 11?

Sometimes we consider continuity over the entire domain of the function. For many functions, this is the entire set of real numbers. A function is continuous on its domain if there are no points of discontinuity in the domain of the function.

Example 12: State where $f(x) = 5x^4 + 6x^2 - 3x + 11$ is continuous.

Applications Involving Limits

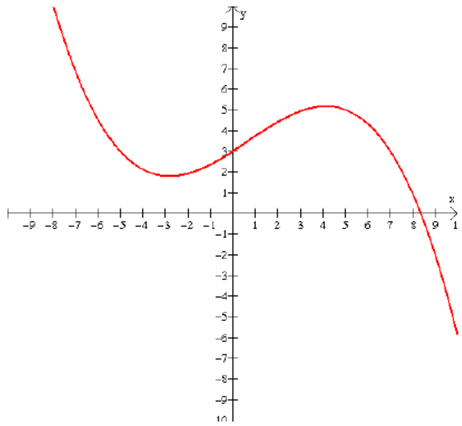
Example 13: The average cost in dollars of constructing each skateboard when x skateboards are produced can be modeled by the function $\overline{C}(x) = 12.5 + \dfrac{123,500}{x}$. What is the average cost per skateboard if the number of boards produced gets larger?

Example 14: The number of adults in America with internet access at home can be modeled by the function $f(t) = \dfrac{180.58}{1 + 2.98e^{-.549t}}$, where $f(t)$ is given in millions of people. In the long run, about how many adults will have internet access if we use this model?

Example 15: Suppose a city discovered the presence of a toxic pollutant in its water treatment facility. The cost of removing x percent off the pollutant, where cost is measured in millions of dollars, is given by $C(x) = \dfrac{0.53x}{100 - x}$, where 0 < x < 100. Find the cost or removing 50%, 60%, 70%, 80%, 90%, 95%, 98% and 99% of the pollutant.

Lesson 6: The Limit Definition of the Derivative; Rules for Finding Derivatives

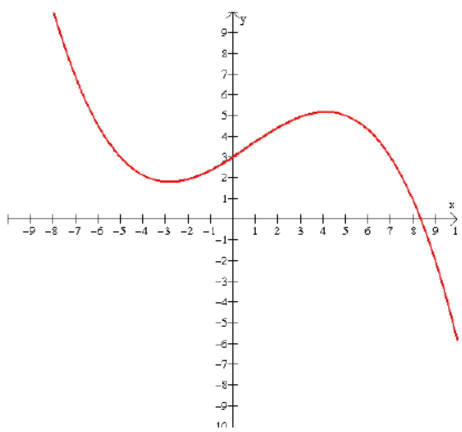
We are interested in finding the slope of the tangent line at a specific point.



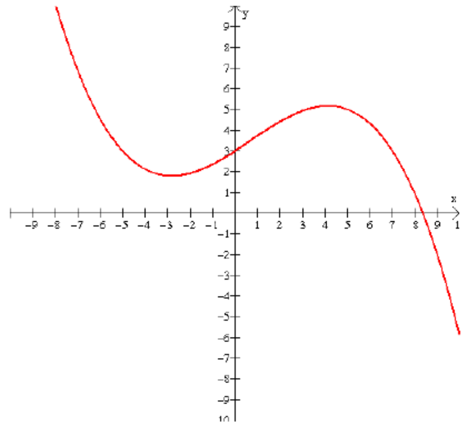
We need a way to find the slope of the tangent line analytically for every problem that will be exact every time.

We can draw a secant line across the curve, then take the coordinates of the two points on the curve, *P* and *Q*, and use the slope formula to approximate the slope of the tangent line.

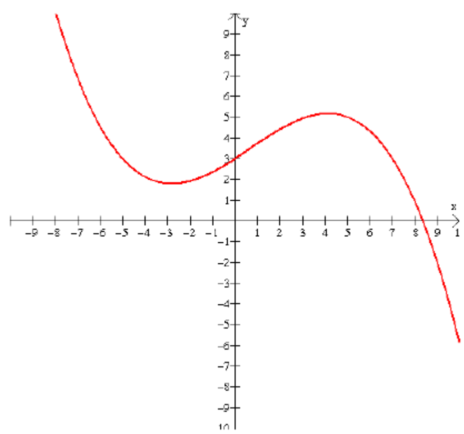
Consider this function:



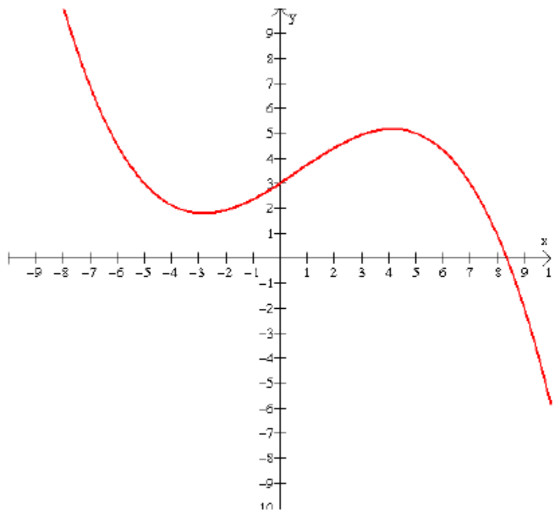
Now suppose we move point *Q* closer to point *P*. When we do this, we'll get a better approximation of the slope of the tangent line.



Suppose we move point *Q* even closer to point *P*. We get an even better approximation. We are letting the distance between *P* and *Q* get smaller and smaller. What does this sound like?



Now let's give these two points names. We'll express them as ordered pairs.



Now we'll apply the slope formula to these two points.

This expression is called a **difference quotient**.

The last thing that we want to do is to let the distance between P and Q get arbitrarily small, so we'll take a limit.

This gives us the definition for the **slope of the tangent line**.

Definition: The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

This limit is also called the **derivative**.

The **derivative of *f* with respect to *x*** is the function f' (read "*f* prime") defined by

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. The domain of $f'(x)$ is the set of all x for which the limit exists.

The difference quotient gives us the **average rate of change.** We find the **instantaneous rate of change** when we take the limit of the difference quotient.

The Four-Step Process for Finding the Derivative

Now that we know what the derivative is, we need to be able to find the derivative of a function. We'll use an algebraic process to do so. We'll use a 4-Step process to find the derivative. The steps are as follows:

1. Find $f(x+h)$.
2. Find $f(x+h)-f(x)$.
3. Form the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.
4. Find the limit of the difference quotient as h gets close to 0: $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

So, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Example 1: Find the rule for the slope of the tangent line for the function $f(x)=5x-3$.

Step 1:

Step 2:

Step 3:

Step 4:

Example 2: Use the four-step process to find the derivative of $f(x) = 3x^2$.

Step 1:

Step 2:

Step 3:

Step 4:

Example 3: Find the derivative of $f(x) = -2x^2 + 6x + 3$. Then find $f'(-2)$.

Example 4: Use the Four-Step Process for finding the derivative of the function $f(x) = \frac{4}{x}$. Then find $f'(1)$.

We can use the ***average rate of change*** to approximate the slope of the tangent line. The average rate of change is given by the difference quotient, $\frac{f(x+h) - f(x)}{h}$. We can also write this formulas as $\frac{f(b) - f(a)}{b - a}$ where a and b are the endpoints of a closed interval [a, b].

Example 5: Suppose $f(x) = 3x^2 - 4x - 1$. Find the average rate of change of f over the interval [2, 5].

Example 6: Suppose the distance covered by a car can be measure by the function $s(t) = 4t^2 + 32t$, where $s(t)$ is given in feet and t is measured in seconds.

A. Find the average velocity of the car over the interval [0, 4].
B. Find the instantaneous velocity of the car when $t = 4$.

Rules for Finding Derivatives

We can use the limit definition of the derivative to find the derivative of every function, but it isn't always convenient. Fortunately, there are some rules for finding derivatives which will make this easier.

First, a bit of notation:

$\frac{d}{dx}\left[f(x)\right]$ is a notation that means "the derivative of *f* with respect to *x*, evaluated at *x*."

Rule 1: The Derivative of a Constant

$\frac{d}{dx}\left[c\right] = 0,$ where *c* is a constant.

Example 1: If $f(x) = 12,$ find $f'(x)$.

Rule 2: The Power Rule

$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ for any real number n

Example 2: If $f(x) = x^4$, find $f'(x)$.

Example 3: If $f(x) = \sqrt[3]{x}$, find $f'(x)$.

Example 4: If $f(x) = \frac{1}{x^5}$, find $f'(x)$.

Rule 3: Derivative of a Constant Multiple of a Function

$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$ where c is any real number

Example 5: If $f(x) = 6x^3$, find $f'(x)$.

Example 6: If $f(x) = -3\sqrt{x}$, find $f'(x)$.

Rule 4: The Sum/Difference Rule

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Example 7: Find the derivative: $f(x) = 5x^4 + 3x^3 - 4 - \frac{6}{x}$.

Example 8: Find the derivative: $f(x) = 4x^3 - 7x^2 + 8x - 5$

Note, there are many other rules for finding derivatives "by hand." We will not be using those in this course. Instead, we will use GeoGebra for finding more complicated derivatives.