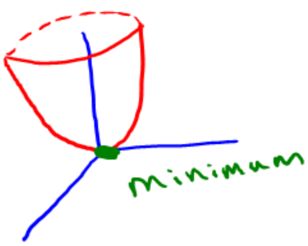
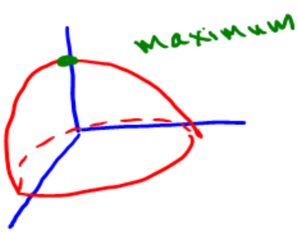
Math 1314 Optimizing

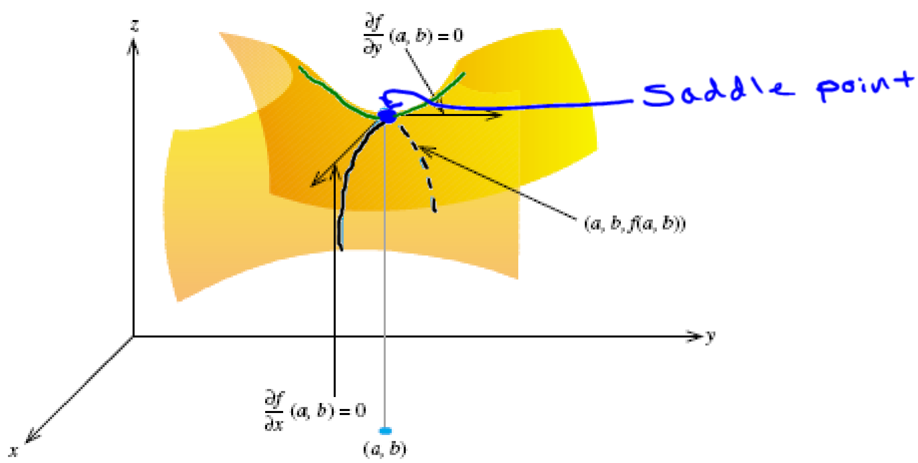
Lesson 24: Maxima and Minima of Functions of ~~Several~~ Two Variables

We learned to find the maxima and minima of a function of a single variable earlier in the course. Although we did not use it much, we had a second derivative test to determine whether a critical point of a function of a single variable generated a maximum or a minimum, or possibly that the test was not conclusive at that point. We will use a similar technique to find relative extrema of a function of several variables.

Since the graphs of these functions are more complicated, determining relative extrema is also more complicated. At a specific critical number, we can have a max, a min, or something else. That "something else" is called a saddle point.







The method for finding relative extrema is very similar to what you did earlier in the course.

First, find the 1 first partial derivatives and 2 set them equal to zero. You will have a system of equations in two variables which you will need to 3 solve to find the critical points.

Second, you will apply the second derivative test. To do this, you must find the 4 second-order partial derivatives. 5 Let $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$. You will compute $D(a,b)$ for each critical point (a,b). Then you can apply **the second derivative test for functions of two variables**:

- If $D(a, b) > 0$ and $f_{xx}(a,b) < 0$, then f has a <u>relative maximum</u> at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a,b) > 0$, then f has a <u>relative minimum</u> at (a, b).
- If $D(a, b) < 0$, then f has neither a relative maximum nor a relative minimum at (a, b) (i.e., it has a saddle point, which is neither a max nor a min).
- If $D(a, b) = 0$, then this test is inconclusive.

6 Find relative extrema

Example 1: Find the relative extrema of the function $f(x,y) = x^2 + 2xy + 2y^2 - 4x + 8y - 1$.

① Find f_x and f_y

Derivative[f,x]

Derivative[f,y]

$f_x = a(x,y) = 2x + 2y - 4$

$f_y = b(x,y) = 2x + 4y + 8$

② Set these = to 0

type in

$2x+2y-4=0$

$2x+4y+8=0$

c: $x+y=2$

d: $x+2y=-4$

③ Find point of intersection

Intersect[c,d]

$(8,-6)$

critical point

④ Find second order partials

Derivative[a,x] $\quad f_{xx} = e(x,y) = 2$

Derivative[a,y] $\quad f_{xy} = g(x,y) = 2$

Derivative[b,y] $\quad f_{yy} = h(x,y) = 4$

Derivative[b,x] $\quad f_{yx} = i(x,y) = 2$

⑤ $D(a,b) = f_{xx} \cdot f_{yy} - [f_{xy}]^2$

$D(8,-6) = 2 \cdot 4 - 2^2 = 8 - 4 = 4 > 0$

$f_{xx} = 2 > 0$

Look at the 4 bullets

We have a relative minimum at $(8,-6)$

⑥ Find minimum value

$f(8,-6)$ $\quad -41$

$\boxed{\text{rel. min value} = -41}$

Example 2: Find the relative extrema of the function $f(x,y) = 2x^3 + y^2 - 9x^2 - 4y + 12x - 2$.

Using GGB:

1. **Find first partial derivatives.**

Derivative [f, x] $\quad f_x = a(x,y) = 6x^2 - 18x + 12$

Derivative [f, y] $\quad f_y = b(x,y) = 2y - 4$

2. **Set first partials equal to zero. Type them into GGB input line.**

$6x^2 - 18x + 12 = 0$ $\quad$ c: $x^2 - 3x = -2$

$2y - 4 = 0$ $\quad$ d: $y = 2$

3. **Find point(s) of intersection. This/these is/are the critical point(s).**

Intersect [c, d] $\quad (1,2) \quad (2,2)$

2 critical points !!

4. **Find second order partial derivatives.**

Derivative [a, x] $\quad f_{xx} = e(x,y) = 12x - 18$

Derivative [a, y] $\quad f_{xy} = g(x,y) = 0$

Derivative [b, y] $\quad f_{yy} = h(x,y) = 2$

Derivative [b, x] $\quad f_{yx} = i(x,y) = 0$

5. **Find D for each critical point. Use the second derivative test information to classify each critical point.**

(1,2) $\quad f_{xx}|_{(1,2)} = 12(1) - 18 = -6 < 0$

$D(1,2) = -6 * 2 - 0^2 = -12 < 0$

saddle point at (1,2)

(2,2) $\quad f_{xx}|_{(2,2)} = 12(2) - 18 = 6 > 0$

$D(2,2) = 6 * 2 - 0^2 = 12 > 0$

relative minimum at (2,2)

6. **Determine any relative extrema of the function.**

$f(2,2) = -2$

$\boxed{\text{relative minimum value} = -2}$

Example 3: Find the relative extrema of the function $f(x, y) = -3x^2 + 2xy - 2y^2 + 14x + 2y - 8$.

① Derivative [f, x] $\quad f_x = a(x, y) = -6x + 2y + 14$

Derivative [f, y] $\quad f_y = b(x, y) = 2x - 4y + 2$

② $-6x + 2y + 14 = 0$ $\quad$ c: $-3x + y = -7$

$2x - 4y + 2 = 0$ $\quad$ d: $x - 2y = -1$

③ Intersect [c, d] $\quad (3, 2)$

critical point

④ Derivative [a, x] $\quad f_{xx} = e(x, y) = -6$

Derivative [a, y] $\quad f_{xy} = g(x, y) = 2$

Derivative [b, y] $\quad f_{yy} = h(x, y) = -4$

Derivative [b, x] $\quad f_{yx} = i(x, y) = 2$

⑤ $D(a, b) = f_{xx} \cdot f_{yy} - [f_{xy}]^2$

$D(3, 2) = -6 \cdot -4 - 2^2 = 24 - 4 = 20 > 0$

$f_{xx} = -6 < 0$

$\boxed{\text{Relative max at } (3, 2)}$

⑥ $f(3, 2) = 3$

$\boxed{\text{Relative max value} = 3}$

$-x^3 + 2xy - y^2 - 5$

Example 4: Find the relative extrema of the function $f(x, y) = \cancel{x^3 - 2xy + y^2 + 5}$.

saddle point @ $(0, 0)$

relative max @ $\left(\frac{2}{3}, \frac{2}{3}\right)$

Example 5: Suppose a company's weekly profits can be modeled by the function $P(x, y) = -0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000$ where profits are given in thousand dollars and x and y denote the number of standard items and the number of deluxe items, respectively, that the company will produce and sell. How many of each type of item should be manufactured each week to maximize profit? What is the maximum profit that is realizable in this situation?

① Derivative [P, x]
Derivative [P, y]

$P_x = a(x,y) = \frac{-2x - y + 500}{5}$

$P_y = b(x,y) = \frac{-2x - 5y + 900}{10}$

② $\frac{-2x - y + 500}{5} = 0$

$\frac{-2x - 5y + 900}{10} = 0$

c: $-0.4x - 0.2y = -100$

d: $-0.2x - 0.5y = -90$

③ Intersect [c, d]

$\boxed{(200, 100)}$

④ Derivative [a, x]
Derivative [a, y]
Derivative [b, y]
Derivative [b, x]

$f_{xx} = e(x,y) = -\frac{2}{5}$

$f_{xy} = g(x,y) = -\frac{1}{5}$

$f_{yy} = h(x,y) = -\frac{1}{2}$

$f_{yx} = i(x,y) = -\frac{1}{5}$

⑤ $D(200, 100) = -\frac{2}{5} \cdot -\frac{1}{2} - \left(-\frac{1}{5}\right)^2 = .16 > 0$

$f_{xx} = -\frac{2}{5} < 0$

relative maximum @ (200, 100)

⑥ Answer the questions

$\boxed{\text{200 standard, 100 deluxe}}$

$P(200, 100) = 10500$

$\boxed{\$10500 * 1000 = \$10{,}500{,}000}$