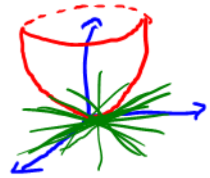
$g(x) = 4$
$g'(x) = 0$

$f(x) = 4x^2 - 8x + 3$
$f'(x) = 8x^1 - 8 = \boxed{8x-8}$

OR
GGB



Math 1314
Lesson 23: Partial Derivatives

When we are asked to find the derivative of a function of a single variable, $f(x)$, we know exactly what to do. However, when we have a function of two variables, there is some ambiguity. With a function of two variables, we can find the slope of the tangent line at a point P from an infinite number of directions. We will only consider two directions, either parallel to the x axis or parallel to the y axis. When we do this, we fix one of the variables. Then we can find the derivative with respect to the other variable. treat it like a constant

So, if we fix y, we can find the derivative of the function with respect to the variable x. And if we fix x, we can find the derivative of the function with respect to the variable y.

These derivatives are called **partial derivatives**.

First Partial Derivatives

Notation:

delta

$f_x = \dfrac{\partial f}{\partial x}$ = first partial with respect to x

"f sub x" y is a constant

$f_y = \dfrac{\partial f}{\partial y}$ = first partial with respect to y

"f sub y"

Example 1: Find the first partial derivatives of the function without using GGB.
$f(x, y) = x^2 - 3xy^2 + 4y^2$.

$\underline{\underline{f_x}}$ treat y like a constant

$f(x,y) = x^2 - 3y^2 \cdot x + \overset{\text{constant}}{4y^2}$

$f_x = 2x - 3y^2 + 0 = \boxed{2x - 3y^2}$

$\underline{\underline{f_y}}$ treat x like a constant

$f(x,y) = \overset{\text{constant}}{x^2} - 3x \cdot y^2 + 4y^2$

$f_y = 0 - 3x \cdot 2y + 8y = \boxed{-6xy + 8y}$

Example 2: Find the first partial derivatives of the function
$f(x, y) = 5x^2y^2 - 2x^3y + 9x^2 - 14y^2 + 10$.

GGB Command:

Derivative [function, variable]

f_x Derivative [f, x]

$$\boxed{f_x = a(x,y) = -6x^2y + 10xy^2 + 18x}$$

f_y

$$\boxed{f_y = b(x,y) = -2x^3 + 10x^2y - 28y}$$

Example 3: Find the first partial derivatives of the function
$f(x, y) = 4x^3y^2 + 2x^2y^3 - 12x^2 + 3y^2 + 10$.

GGB Command:

Derivative [function, variable]

f_x $f_x = 12x^2y^2 + 4xy^3 - 24x$

Derivative [f, x]

f_y $f_y = 8x^3y + 6x^2y^2 + 6y$

Derivative [f, y]

we found partials in ex 3.

Example 4: Find the first partial derivatives of the function $f(x,y)=4x^3y^2+2x^2y^3-12x^2+3y^2+10$ evaluated at the point (-1, 3).

$\to f_y\big|_{(-1,3)}$

Notation: $f_x\big|_{(-1,3)}$ $f_y\big|_{(-1,3)}$

GGB Command:

for $f_x\big|_{(-1,3)} \to$ a(-1,3) ; for $f_y\big|_{(-1,3)} \to$ b(-1,3)

$f_x\big|_{(-1,3)} = 24$

a(-1,3)

$f_y\big|_{(-1,3)} = 48$

b(-1,3)

Second-Order Partial Derivatives

Sometimes we will need to find the second-order partial derivatives. To find a second-order partial derivative, you will take respective partial derivatives of the first partial derivative. There are a total of 4 second-order partial derivatives.

Notation:

f_x f_y

1st f_{xx} 2nd

1st f_{xy} 2nd

1st f_{yy} 2nd

1st f_{yx} 2nd

4 second order partials!!

Example 5: Find the second-order partial derivatives of the function $f(x, y) = 3x^2y^2 - 5x^2 + 10y$.

GGB Command:

$f_x = a(x,y) = 6xy^2 - 10x$

$f_y = b(x,y) = 6x^2y + 10$

a $\quad f_{xx} = c(x,y) = 6y^2 - 10$ Deriv[a, x]

a $\quad f_{xy} = 12xy$ Deriv[a, y]

b $\quad f_{yy} = 6x^2$ Deriv[b, y]

b $\quad f_{yx} = 12xy$ Deriv[b, x]

What do you notice about the mixed partials? $f_{xy} = f_{yx}$

they are equal

Example 6: Find the second-order partial derivatives of the function $f(x, y) = 3x^2 - x^3y^3 + 5xy + 6y^3$.

GGB Command:

$f_x = a(x,y) = -3x^2y^3 + 6x + 5y$

$f_y = b(x,y) = -3x^3y^2 + 5x + 18y^2$

a $f_{xx} = c(x,y) = -6xy^3 + 6$

a $f_{xy} = d(x,y) = -9x^2y^2 + 5$

b $f_{yy} = e(x,y) = -6x^3y + 36y$

b $f_{yx} = g(x,y) = -9x^2y^2 + 5$

What do you notice about the mixed partials?

Example 7: Evaluate the first and second-order partial derivatives of $f(x, y) = 3x^2 - x^3y^3 + 5xy + 6y^3$ at the point (1, 2).

GGB Command:

function name (1, 2)

$f_x\big|_{(1,2)} = a(1,2) = -8$

$f_y\big|_{(1,2)} = b(1,2) = 65$

$f_{xx}\big|_{(1,2)} = c(1,2) = -42$

$f_{xy}\big|_{(1,2)} = d(1,2) = -31$

$f_{yy}\big|_{(1,2)} = e(1,2) = 60$

$f_{yx}\big|_{(1,2)} = g(1,2) = -31$

f(x,y)

Derivative [f, x] $f_x = a(x,y)$

Derivative [f, y] $f_y = b(x,y)$

2nd derivatives

f_{xx} = Deriv [a, x] f_{yy} = Der [b,y]

f_{xy} = Deriv [a,y] = f_{yx} = Deriv [b,x]

A function of the form $f(x, y) = ax^b y^{1-b}$ where a and b are positive constants and $0 < b < 1$ is called a ***Cobb-Douglas production function***. In this function, x represents the amount of money spent for labor, and y represents the amount of money spent on capital expenditures such as factories, equipment, machinery, tools, etc. The function measures the output of finished products.

sum of exp. = 1

The first partial with respect to x is called the marginal productivity of labor. It measures the change in productivity with respect to the amount of money spent for labor. In finding the first partial with respect to x, the amount of money spent on capital is held at a constant level.

The first partial with respect to y is called the marginal productivity of capital. It measures the change in productivity with respect to the amount of money spent on capital expenditures. In finding the first partial with respect to y, the amount of money spent on labor is held at a constant level.

Example 8: A country's production can be modeled by the function $f(x, y) = 50x^{2/3}y^{1/3}$ where x gives the units of labor that are used and y represents the units of capital that were used.

enter f into GGB

A. Find the first partial derivatives.

Derivative [f, x] $f_x = a(x,y) = \dfrac{\frac{100}{3}\sqrt[3]{y}}{\sqrt[3]{x}}$

Derivative [f, y] $f_y = b(x,y) = \dfrac{\frac{50}{3}\sqrt[3]{x^2}}{\sqrt[3]{y^2}}$

B. Find the marginal productivity of labor and the marginal productivity of capital when the amount expended on labor is 125 units and the amount spent on capital is 27 units.

$a(125, 27) = 20$ increasing at the rate of 20 units / unit of labor expense

$b(125, 27) = 46.2963$ increasing at the rate of 46.2963 units / unit of capital expence

C. Should the government of the county encourage capital investment or labor investment?

$\boxed{\text{Capital investment}}$

$\sqrt[3]{x^2}$

$(x \wedge 2) \wedge (1/3)$

Example 9: A company's revenues can be modeled by the function $R(x,y) = -0.2x^2 - 0.25y^2 - 0.2xy + 200x + 160y$ where x gives the number of product A and y gives the number of product B that are produced and sold each week and $R(x,y)$ gives revenues ~~in thousands of~~ dollars. Find the first partial derivatives and evaluate them when $x = 300$ and $y = 250$. Explain the results.

enter R(x,y) into GGB

Derivative[R, x]

Derivative[R, y]

$R_x = a(x,y) = \frac{-2x - y + 1000}{5}$

$R_y = b(x,y) = \frac{-2x - 5y + 1600}{10}$

$R_x \big|_{(300,250)} = a(300,250) = 30$

increasing at the rate of \$30/ unit increase in A

$R_y \big|_{(300,250)} = b(300,250) = -25$

decreasing at the rate of \$25/unit increase in B