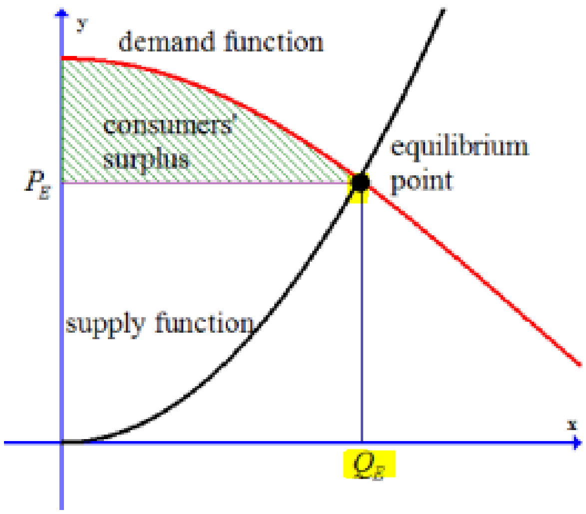
**Math 1314**
**Lesson 21: Other Applications of Integration**

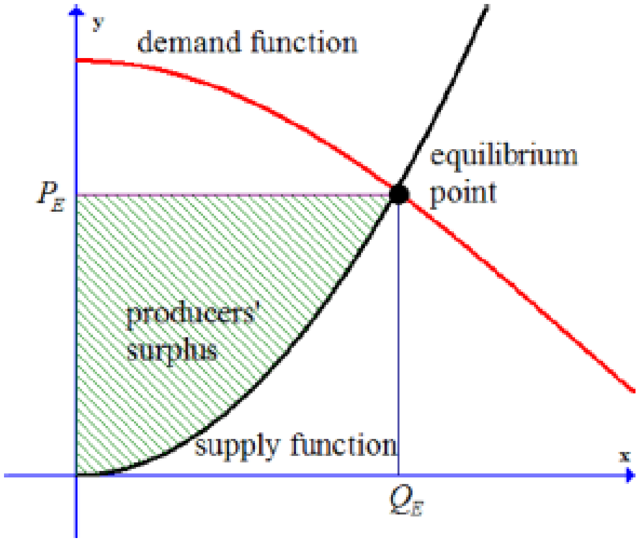
## Consumers' Surplus and Producers' Surplus

The **consumers' surplus** is defined to be the difference between what customers would be willing to pay and what they actually pay. It is the area of the region bounded above by the demand function and below by the line that represents the unit market price. In the sketch shown below, the shaded region represents the consumers' surplus.




Then the consumers' surplus is given by $CS = \int_0^{Q_E} D(x)\,dx - \overbrace{Q_E \cdot P_E}^{\text{area of rectangle}}$. In this formula, $Q_E$ represents the quantity sold and $P_E$ represents the price.

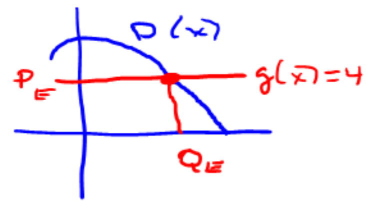
Similarly, producers may be willing to sell their product for a lower price than the prevailing market price. If the market price is higher than where producers expect to price their items, then the difference is called the **producers' surplus**. In the sketch shown below, the shaded region represents the producers' surplus.




The producers' surplus is given by $PS = Q_E \cdot P_E - \int_0^{Q_E} S(x)\,dx$ where $S(x)$ is the supply function, $P_E$ represents the unit market price and $Q_E$ represents the quantity supplied. The producer's surplus is the area of the region bounded above by the line that represents the price and below by the supply curve.

$D(x)$

**Example 1:** Suppose the demand for a certain product is given by $p = -0.01x^2 - 0.1x + 6$ where $p$ is the unit price given in dollars and $x$ is the quantity demanded per month given in units of 1000. The market price for the product is \$4 per unit.




A. Find the quantity demanded at the given price.

Intersect [0, 9]

~~(-20, 4)~~

(10, 4)

$Q_E$, $P_E$

$x = 10$

(10, 000)

B. Find the consumers' surplus if the market price for the product is \$4 per unit.

$$CS = \int_0^{10} (-0.01x^2 - 0.1x + 6)\,dx - 10 * 4$$

$51.6667 - 40 = 11.6667$

Now adjust $11.6667 * 1000 = 11666.7$

$\boxed{\$11667}$

Sometimes, the unit price will not be given. Instead, product will be sold at market price, and you'll be given both supply and demand equations. In this case, we can find the equilibrium point (Section 4.2) which will give us the equilibrium quantity and price.

**Example 2**: The demand function for a popular make of 12-speed bicycle is given by $p = D(x) = -0.001x^2 + 250$ where $p$ is the unit price in dollars and $x$ is the quantity demanded in units of a thousand. The supply function for the same product is given by

$p = S(x) = 0.0006x^2 + 0.02x + 100$ where $p$ is the unit price in dollars and $x$ is the quantity supplied in units of a thousand. Determine the consumers' surplus and the producers' surplus if the market price is set at the equilibrium price.

§ 4.2

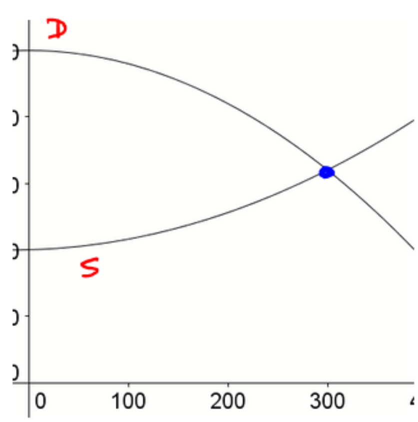
**Point of Intersection:**

D = S

pt. of intersect.

~~(-317.5, 152.3438)~~

(300, 160)




**What does the point of intersection represent?**

x = equilibrium quantity

y = equilibrium price

**CS Formula:** $CS = \int_0^{Q_E} D(x)\,dx - Q_E * P_E$

**CS:** $CS = \int_0^{300} (-0.001x^2 + 250)\,dx - 300 * 160$

Integral[D, 0, 300] - 300 * 160 = 12000

$\boxed{\$12{,}000{,}000}$

**PS Formula:**

$PS = Q_E * P_E - \int_0^{Q_E} S(x)\,dx$

**PS:**

$PS = 300 * 160 - \int_0^{300} (0.0006x^2 + 0.02x + 100)\,dx$

300*160 - Integral[S, 0, 300] = 11700

11700*1000

$\boxed{\$11{,}700{,}000}$

**Probability**

The study of probability deals with the likelihood of a certain outcome of an experiment. When you flip a coin, the probability that the coin lands with heads facing up is $\frac{1}{2}$. When you roll a six-sided die and record the number that lands on the uppermost face, the probability that the die lands with the number 3 facing upwards is $\frac{1}{6}$.

You can also find the probability of something occurring over a continuous interval. Suppose you want to know how long a brand of light bulb lasts. Now the possible set of answers contains more than a discrete set of numbers (e.g., something other than 1, 2, 3, etc.). The light bulbs can last any positive length of time. We can find the probability that the light bulbs life span is on a given interval [a, b].

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This problem is an example of a problem that involves a probability density function. This type of function can be used to determine the probability that an event occurs on a given interval [a, b]. All probability density functions must meet these three criteria:


1. $f(x) \geq 0$ for all $x$
2. The area under the graph of $f(x)$ is exactly 1.
3. The probability that an event occurs in an event [a, b] can be computed using the definite integral $\int_a^b f(x)\,dx$.

**Example 3:** The function $f(x) = 0.002e^{-0.002x}$ gives the life span of a popular brand of light bulb, where $x$ gives the lifespan in hours and $f(x)$ is the probability density function. Find each probability:

↳ gives probability

A. Find the probability that the lifespan is between 500 hours and 1000 hours. [500, 1000]

$$\int_{500}^{1000} (0.002e^{-0.002x})\,dx$$

Integral[f, 500, 1000] = $\boxed{.2325}$

B. Find the probability that the lifespan is between 1500 hours and 2000 hours. [1500, 2000]

$$\int_{1500}^{2000} (0.002e^{-0.002x})\,dx = \boxed{.0315}$$

**Example 4:** A company finds that the percent of its locations that experience a profit in the first year of business has the probability density function $P(x) = \frac{36}{11}x\left(1 - \frac{1}{3}x\right)^2$, $0 \le x \le 1$.

P(x) = (36/11) x*(1 − (1/3) x) ^2

A. What is the probability that more than 50% of the company's locations experienced a profit during the first year of business? Set up the needed integral and then use it to answer the question.

[50, 100] → [.5, 1]

$$\int_{.5}^{1} \left[\frac{36}{11}x\left(1 - \frac{1}{3}x\right)^2\right] dx = \boxed{.6761}$$

B. What is the probability that between 30% and 60% of the company's locations experienced a profit during the first year of business? Set up the needed integral and then use it to answer the question.

[30, 60] → [.3, .6]

$$\int_{.3}^{.6} \left[\frac{36}{11}x\left(1 - \frac{1}{3}x\right)^2\right] dx$$

Integral [P, .3, .6] = $\boxed{.3154}$