Math 1314 ONLINE
Lesson 18: Finding Indefinite and Definite Integrals

Working with Riemann sums can be quite time consuming, and at best we get a good approximation. In an area problem, we want an exact area, not an approximation. The definite integral will give us the exact area, so we need to see how we can find this.

We need to start by finding an antiderivative:

Antiderivatives

Definition: A function F is an antiderivative of f on interval I if $F'(x) = f(x)$ for all x in I.

You can think of an antiderivative problem as asking you to find the *problem* if you are given the *derivative*. Look at what you are given in your problem and ask: "If this is the answer, what was the problem whose derivative I wanted to find?"

Antiderivative problems will use this notation:

$$F(x) + C = \int f(x)\,dx$$

In this notation, C is an arbitrary constant.

Why do we need a "+ C" when finding antiderivatives?

Remember that the derivative of any constant is zero, so when we take the derivative of $F(x)+C$, we'll get $F'(x)$. If we are starting with $f(x)$ and working backwards to find the antiderivative, we must allow for the possibility that the function had a constant term. In each example given below, the derivative of the function is 5.

$$G(x) = 5x + 2$$
$$H(x) = 5x - 9$$
$$K(x) = 5x - 20000$$

So which of these functions would be the answer for $\int 5\,dx$? We don't know. But we can account for an possible constant as part of the antiderivative by adding C any time we find an antiderivative.

Note that antiderivatives are also called *indefinite integrals.*

what is this the derivative of?

indefinite integral

$\int 2x\,dx$

$2\int x\,dx$

$\cancel{2}\cdot\frac{x^2}{\cancel{2}} + C = x^2 + C$

We have some rules for finding antiderivatives:

Rule 1: The Indefinite Integral of a Constant

$\int k\,dx = kx + C$

Example 1: Find the antiderivative:

$\int 5\,dx$

what function am I the deriv. of

$\boxed{5x + c}$

Rule 2: The Power Rule

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

Example 2: Find the antiderivative:

$\int x^4\,dx$

what function am I the derivative of?

$\frac{x^5}{5} + c$

$\boxed{\frac{1}{5}x^5 + c}$

ck.

$\frac{d}{dx}\left[\frac{1}{5}x^5 + c\right]$

$\frac{1}{5} \cdot 5x^4 + 0$

x^4 ✓

Example 3: Find the antiderivative:

$\sqrt{a} = a^{1/2}$

$$\int \sqrt{x}\ dx$$

$\int x^{1/2}\, dx$

$\dfrac{x^{1/2+1}}{1/2+1} + C$

$\dfrac{x^{3/2}}{3/2} + C$

$\div \frac{3}{2} \Rightarrow \cdot \frac{2}{3}$

$\frac{2}{3} x^{3/2} + C$

Example 4: Find the antiderivative:

$$\int \frac{1}{x^4} dx = \int x^{-4}\, dx$$

$\dfrac{x^{-4+1}}{-4+1} + C$

$-\frac{1}{3}x^{-3} + C$

x^{-4}

$\frac{1}{x^4}$

$\dfrac{x^{-3}}{-3} + C$

$\boxed{\dfrac{1}{-3x^3} + C}$

POPPER 11, question 5

$$\int x^5\ dx$$

A. $5x^4 + C$

B. $\frac{1}{6}x^6 + C$

C. $x^6 + C$

D. $\frac{1}{4}x^4 + C$

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)dx = c\int f(x)dx$$

Example 5: $\int 5x^3 dx = 5\int x^3\,dx$

$\frac{5}{1}\cdot\frac{x^4}{4} + c$

$\boxed{\frac{5}{4}x^4 + c}$

$\int 2x\,dx$

$\frac{2x^2}{2}$ x^c

$x^2 + c$

$\int 5x^3\,dx$

$\frac{5x^4}{4} + c$

$\boxed{\frac{5}{4}x^4 + c}$

Rule 4: The Sum (Difference) Rule

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Example 6: $\int (2x^2 + 5x + 1)dx$

$\int 2x^2\,dx + \int 5x\,dx + \int 1\,dx$

$2\int x^2\,dx + 5\int x^1\,dx + \int 1\,dx$

$2\cdot\frac{x^3}{3} + c_1 + 5\cdot\frac{x^2}{2} + c_2 + 1x + c_3$

$\boxed{\frac{2}{3}x^3 + \frac{5}{2}x^2 + x + c}$

$\int (2x^2 + 5x + 1)\,dx$

$\frac{2x^3}{3} + \frac{5x^2}{2} + x + c$

$\boxed{\frac{2}{3}x^3 + \frac{5}{2}x^2 + x + c}$

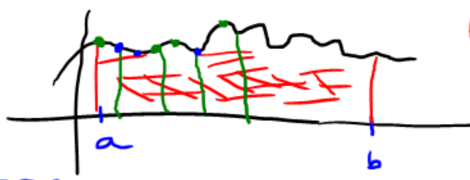
Example 7: $\int (8x^3 - 4x^1 - 3)dx$

$$\frac{8x^4}{4} - \frac{4x^2}{2} - 3x + C$$

$$\boxed{2x^4 - 2x^2 - 3x + c}$$

$f(x) = 3x^2 + 2 \quad [0, 8] \quad n = 4$

Δx intervals   function values approx area



Approx: use Riemann (upper/lower sums)

The Fundamental Theorem of Calculus

Exact area!! Tool is FTOC

Finding the antiderivative is a tool that we need in order to find the definite integral of a function over an interval. Next we apply **the fundamental theorem of calculus**:

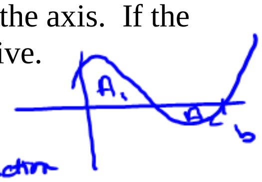
Let f be a continuous function on [a, b]. Then $\int_a^b f(x)dx = F(b) - F(a)$ where $F(x)$ is any antiderivative of f.

This says that we can find the definite integral by first finding the antiderivative of the function that's given and then by evaluating the antiderivative at the upper and lower limits of integration and subtracting.

$\int_0^a f(x)\,dx =$ exact area

If the function is non-negative (never dips below the x-axis) then the definite integral gives the area under the curve on the interval [a, b]. If the function crosses the x axis, so that some of its y values are below the x-axis, then the definite integral gives the "net" of the two areas. Subtract the area of the part that is below the axis from the area of the part that is above the axis. If the area below the axis is larger, it is possible to get a definite integral that is negative.

general function



$\int_0^b f(x)\,dx$

$A_1 - A_2$

Example 8: Evaluate: $\int_1^3 (3x^2 + 4x - 7)dx$

① find antiderative

$$\int_1^3 (3x^2 + 4x^1 - 7)\,dx$$

$$\frac{\cancel{3}x^3}{\cancel{3}} + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} - 7x + c \Big|_1^3$$

$$x^3 + 2x^2 - 7x + c\Big|_1^3$$

$F(b) - F(a)$

② evaluate

$$[3^3 + 2(3)^2 - 7(3) + c] - [1^3 + 2(1)^2 - 7(1) + c]$$

$$[27 + 18 - 21 + c] - [1 + 2 - 7 + c] = [24 + c] - [-4 + c]$$

$$24 + \cancel{c} + 4 - \cancel{c} = \boxed{28}$$

Omit +C for definite integral problems
(but we need +C in indefinite integrals/antiderivs)

Example 9: Evaluate: $\int_1^9 \left(2x+\sqrt{x}\right)dx$

rewrite $\int_1^9 (2x^1 + x^{1/2})\,dx$

$\frac{1}{2}+1=\frac{3}{2}$

antideriv. $\left.\frac{\cancel{2}x^2}{\cancel{2}} + \frac{x^{3/2}}{3/2}\right|_1^9$

$\div \frac{3}{2} \to \cdot \frac{2}{3}$

simplify $\left.x^2 + \frac{2}{3}x^{3/2}\right|_1^9$

evaluate $\left[(9)^2 + \frac{2}{3}(9)^{3/2}\right] - \left[(1)^2 + \frac{2}{3}(1)^{3/2}\right]$

$\left(81 + \frac{2}{3}\cdot 27\right) - \left(1 + \frac{2}{3}\right)$

$(81 + 18) - \left(1\frac{2}{3}\right) = 99 - 1\frac{2}{3} = \boxed{97\frac{1}{3}}$

Example 10: Suppose $f(x) = 2x+1$. Find the area under the graph of f and above the x axis from x = 1 to x = 3.

must graph!!!

set up $\int_1^3 (2x+1)\,dx$

antideriv. $\left.\frac{\cancel{2}x^2}{\cancel{2}} + x\right|_1^3$

simplify $\left.x^2+x\right|_1^3$

evaluate $(3^2+3) - (1^2+1)$

$(9+3) - (1+1)$

$12 - 2 = 10$