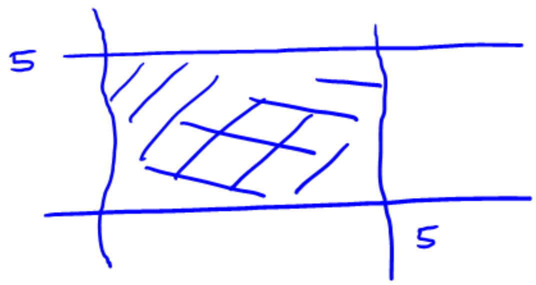
Math 1314
Lesson 16: Area and Riemann Sums

The second question studied in calculus is the area question. If a region conforms to a known formula from geometry, then finding the area is not difficult; simply determine the dimensions and apply the appropriate formula.

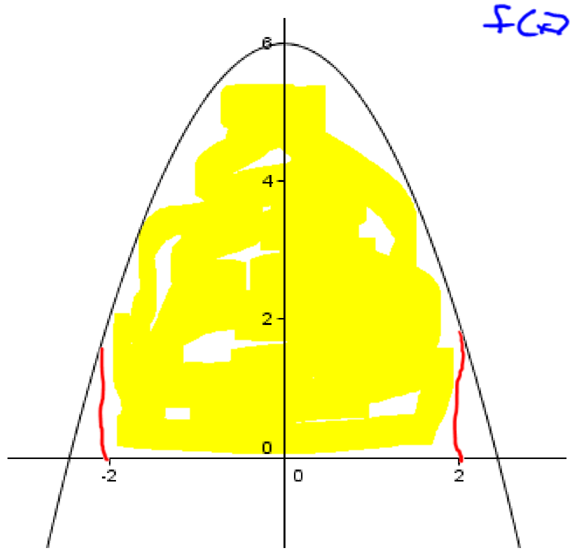
Example 1: Find the area of the region bounded by the x axis, the y axis, the vertical line x = 5 and $f(x) = 5$.



$A = lw$
$A = 5 * 5 = \underline{25}$

Some problems are quite simple to do and do not require calculus. Suppose we want to find the area of a region that is not so nicely shaped. For example, consider the function shown below. The area below the curve and above the *x* axis cannot be determined by a known formula, so we'll need a method for approximating the area.

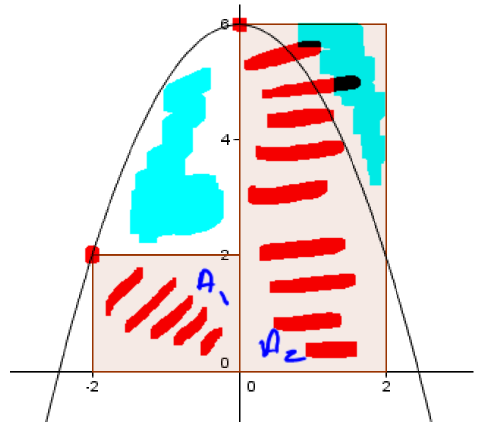
$f(x) = 6 - x^2$



Suppose we want to find the area under the parabola and above the *x* axis, between the lines *x* = 2 and *x* = -2.

We can approximate the area under the curve by subdividing the interval [-2, 2] into smaller intervals and then draw rectangles extending from the *x* axis up to the curve. Suppose we divide the region into two parts and draw two rectangles. We can find the area of each rectangle and add them together. That will give us an approximation of the area under the curve. This method is called "finding a Riemann sum."

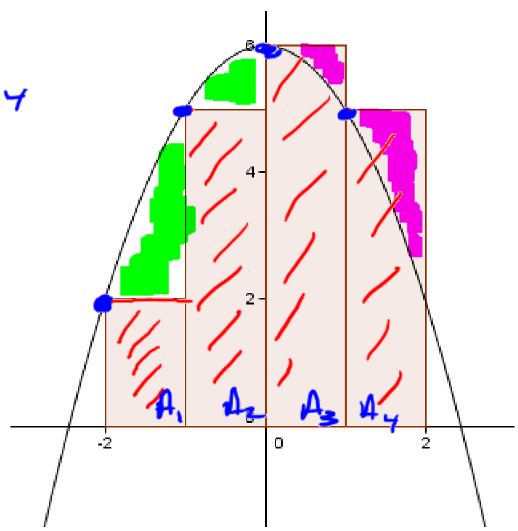
$A \approx A_1 + A_2$



This would not give a very good approximation, as a large region in Quadrant 2 will be left out in the approximation of area, and a large region in Quadrant 1 will be included and should not be. Hopefully we can agree that this is not a very good approximation!

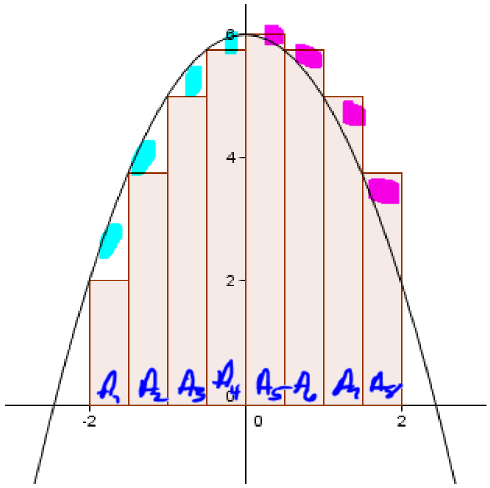
Now, suppose we increase the number of rectangles that we draw to four. We'll find the area of each of the four rectangles and add them up. Here's the graph for this situation.

$A \approx A_1 + A_2 + A_3 + A_4$

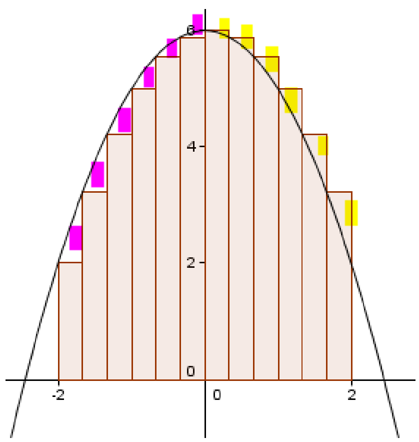


The approximation will be more accurate, but it still isn't perfect. Let's increase the number of rectangles to 8:

$A \approx A_1 + A_2 + \cdots + A_8$

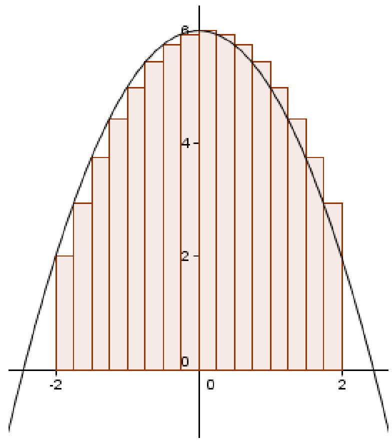


How about 12 rectangles?

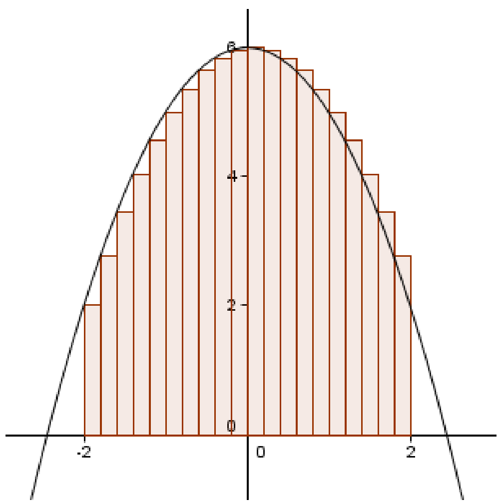


As we increase the number of rectangles, the regions that are included that should be are getting smaller, and the regions that are not included that should be are also getting smaller.

Here's what it looks like with 16 rectangles:



And 20 rectangles:



As we add more and more rectangles, the accuracy improves. We're still not to an exact area, but the area we'd find using 20 rectangles is clearly more accurate than the area we'd find if we just used 2.

Suppose we let the number of rectangles increase without bound. If we do this, the width of each rectangle becomes smaller and smaller, as the number of rectangles approaches infinity, there will be no area that is included that shouldn't be and none left out that should be included. This process is beyond the scope of this course, so we will limit the number of rectangles in the problems we work to a finite number.

Subdivide interval
find area of each rectangle
add them up

Now, how do we approximate the area?

1. **Start by finding the width of each rectangle**. We can compute the width of the rectangles using this formula:

$$\Delta x = \frac{b-a}{n}$$

In this formula, a and b are the endpoints of the interval $[a, b]$ and n is the number of rectangles. Δx stands for "the change in x."

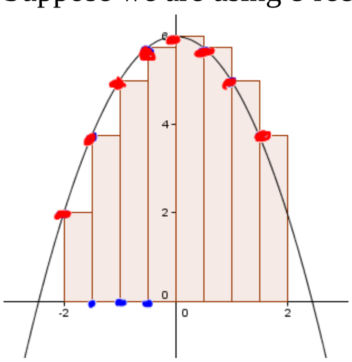
$n=$ a b

Suppose we are using 8 rectangles and the function is $f(x) = 6 - x^2$. The interval is [-2, 2].

$$\Delta x = \frac{2-(-2)}{8} = \frac{4}{8} = \frac{1}{2}$$

2. **Now find the height of the rectangles**. Subdivide the interval into n subintervals, each of width Δx.

Suppose we are using 8 rectangles and the function is $f(x) = 6 - x^2$. The interval is [-2, 2].



Write down the sub intervals

$[-2, -1\frac{1}{2}]$ $[-1\frac{1}{2}, -1]$ $[-1, -\frac{1}{2}]$ $[-\frac{1}{2}, 0]$ $[0, \frac{1}{2}]$ $[\frac{1}{2}, 1]$
$[1, 1\frac{1}{2}]$ $[1\frac{1}{2}, 2]$

Now look at the graph with 8 rectangles and note that the left hand side of each rectangle touches the curve in the graphs. We can compute the value of the function for each of those points.

3. **Find the area of each rectangle.**

-2	2
-1.5	3.75
-1	5
-0.5	5.75
0	6
0.5	5.75
1	5
1.5	3.75

$A_1 = \frac{1}{2}(2) = 1$

$A_2 = \frac{1}{2}(3.75) = 1.875$

$A_3 = \frac{1}{2}(5) = 2.5$

$A_4 = \frac{1}{2}(5.75) = 2.875$

$A_5 = \frac{1}{2}(6) = 3$

$A_6 = \frac{1}{2}(5.75) = 2.875$

$A_7 = \frac{1}{2}(5) = 2.5$

$A_8 = \frac{1}{2}(3.75) = 1.875$

4. **Add them up**. $1 + 1.875 + 2.5 + 2.875 + 3 + 2.875 + 2.5$
$+ 1.875 = \boxed{18.5}$

$$A \approx \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) + \frac{1}{2}f(x_3) + \cdots + \frac{1}{2}f(x_8)$$

factor out $\frac{1}{2}$ $A \approx \frac{1}{2}\left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_8)\right]$

$$A \approx \frac{1}{2}\left[2 + 3.75 + 5 + 5.75 + 6 + 5.75 + 5 + 3.75\right]$$

$$A \approx \frac{1}{2}(37) = \boxed{18.5}$$

This gives an approximation of the area under the curve on the interval [-2, 2] with 8 rectangles using left endpoints.

POPPER 10, questions 6 – 8

Suppose $f(x) = x^2 - 4x + 8$ and you want to approximate the area under the curve on the interval [0, 8] using Riemann sums, 4 rectangles and left endpoints.

6. What is Δx?

$\Delta x = \frac{b-a}{n}$

A. 8 B. 4 C. 2 D. 1 E. 0

7. Which of these is one of the intervals that you'll use to approximate the area?

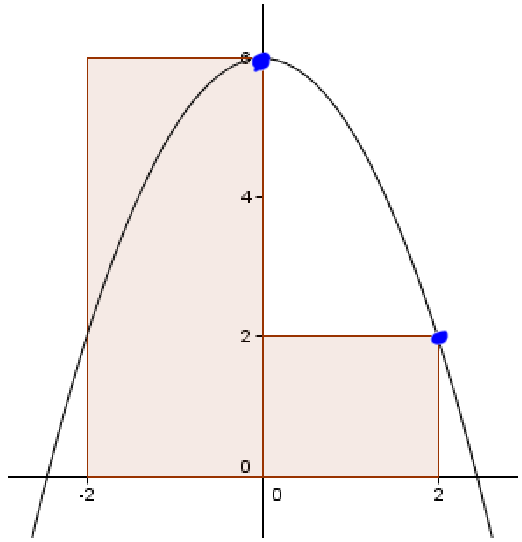
A. [1, 3] B. [0, 4] C. [4, 6] D. [7, 8] E. [2, 6]

8. Find the approximate area.

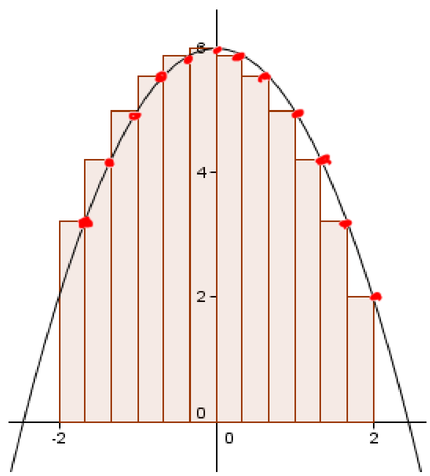
A. 40 B. 20 C. 160 D. 80 E. 100

Using left endpoints is not the only option we have in working these problems. We can also use right endpoints. The graph below shows the region with two rectangles, using right endpoints.

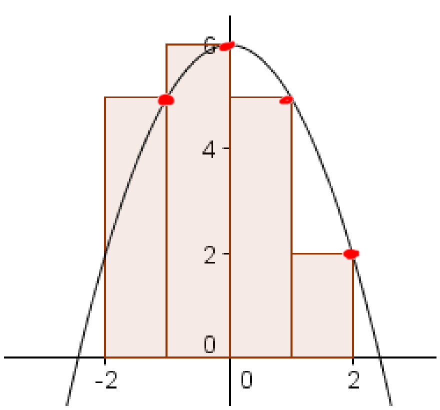
[-2, 0] [0, 2]



The graph below shows the region with 12 rectangles, using right endpoints.



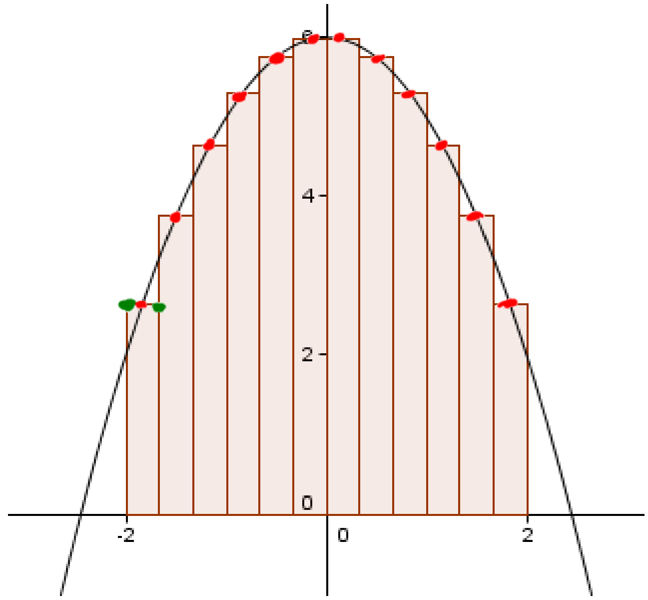
Here's the method for finding the Riemann sum using four rectangles and right endpoints:



$A \approx A_1 + A_2 + A_3 + A_4$

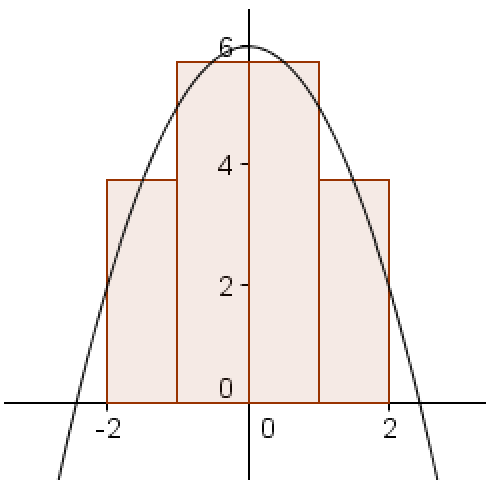
use right endpts.

We can also use the midpoint of each subinterval for finding the height of each rectangle. This graph shows the region with 12 rectangles using midpoints.



We can find the midpoint of each subinterval by averaging the endpoints of the subinterval.

Here’s the method for finding the Riemann sum using four rectangles and midpoints.



We'd like to simplify the arithmetic as much as possible:

To get an exact area, we would need to let the number of rectangles increase without bound:

$$A = \lim_{n\to\infty}\left[f\left(x_1\right) + f\left(x_2\right) + \ldots + f\left(x_n\right)\right]\cdot \Delta x$$

This last computation is quite difficult, we will not work problem of this type. Instead, we will use a limited number of rectangles in the problems that we work.

The process we are using to approximate the area under the curve is called "finding a Riemann sum." These sums are named after the German mathematician who developed them.

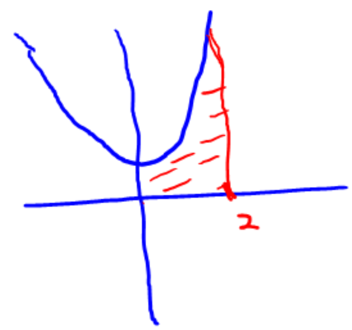
Now let's work some problems:

Example 3: Use left endpoints and 4 subdivisions of the interval to approximate the area under $f(x) = 2x^2 + 1$ on the interval $[0, 2]$.

$4 = n$; $[0, 2]$: a, b

① find Δx

$$\frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$$



② Find subintervals

$$[0, \tfrac{1}{2}]\ [\tfrac{1}{2}, 1]\ [1, \tfrac{3}{2}]\ [\tfrac{3}{2}, 2]$$

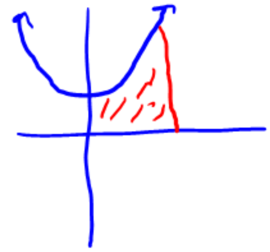
③ find $f(0) = 1$ $\quad f(1) = 3$

$f(\tfrac{1}{2}) = 1.5$ $\quad f(\tfrac{3}{2}) = 5.5$

④ $A \approx \Delta x\left[f(0) + f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2})\right]$

$\approx \frac{1}{2}\left[1 + 1.5 + 3 + 5.5\right] = \frac{1}{2}(11) = \boxed{5.5}$

Example 4: Use right endpoints and 4 subdivisions of the interval to approximate the area under $f(x) = 2x^2 + 1$ on the interval $[0, 2]$.



① $\Delta x = \frac{2-0}{4} = \frac{1}{2}$

② $[0, \frac{1}{2}]$ $[\frac{1}{2}, 1]$ $[1, 1.5]$ $[1.5, 2]$

③ find $f(\frac{1}{2}) = 1.5$ $f(1.5) = 5.5$

$f(1) = 3$ $f(2) = 9$

④ $A \approx \frac{1}{2}[1.5 + 3 + 5.5 + 9] = \frac{1}{2}(19) = \boxed{9.5}$

Example 5: Use midpoints and 4 subdivisions of the interval to approximate the area under $f(x) = 2x^2 + 1$ on the interval $[0, 2]$.

① $\Delta x = \frac{2-0}{4} = \frac{1}{2}$

② $[0, \frac{1}{2}]$ $[\frac{1}{2}, 1]$ $[1, \frac{3}{2}]$ $[\frac{3}{2}, 2]$

midpoints: $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$

$\frac{0+\frac{1}{2}}{2} = \frac{1}{4}$

$\frac{\frac{1}{2}+1}{2} = \frac{\frac{3}{2}}{2}$

$\frac{1+\frac{3}{2}}{2} = \frac{\frac{5}{2}}{2}$

$\frac{\frac{3}{2}+2}{2} = \frac{\frac{7}{2}}{2}$

③ $f(\frac{1}{4}) = 1.125$ $f(\frac{5}{4}) = 4.125$

$f(\frac{3}{4}) = 2.125$ $f(\frac{7}{4}) = 7.125$

④ $A \approx \frac{1}{2}[1.125 + 2.125 + 4.125 + 7.125]$

$A \approx \frac{1}{2}(14.5) = \boxed{7.25}$