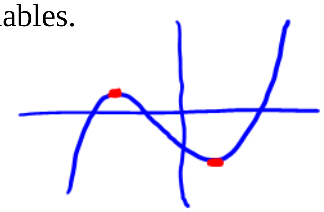
as big as possible
or as small as possible

Math 1314
Lesson 15: Second Derivative Test and Optimization

The Second Derivative Test

but not in this class

There is a second derivative test to find relative extrema. It is sometimes convenient to use; however, it can be inconclusive. Later in the course, we will use a similar second derivative test to find maxima and minima of functions with two variables.



The Second Derivative Test:

$f'(x) = 0$ solve

1. Find all critical numbers.
2. Compute $f''(c)$ for each critical number c.
   (a) If $f''(c) > 0$, then f has a relative minimum at c. rel min
   (b) If $f''(c) < 0$, then f has a relative maximum at c. rel max
   (c) If $f''(c) = 0$, then the test fails. It is inconclusive. Try another method.

Note: to find the y coordinate of a relative maximum or relative minimum, substitute the value you found for x into the original function.

Example 1: Find any relative extrema using the Second Derivative Test if

$f(x) = \frac{1}{3}x^3 - 2x^2 - 5x - 10.$

① $f'(x) = x^2 - 4x - 5$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x - 5 = 0 \quad x + 1 = 0$

c.n. → $x = 5 \quad x = -1$

② $f''(x) = 2x - 4$

$f''(5) = 2(5) - 4 = 10 - 4 = 6$ ∪

$(5, -43.3333)$ rel min

$f''(-1) = 2(-1) - 4 = -2 - 4 = -6$ ∩

rel max

$(-1, -7.3333)$

Example 2: Find any extrema using the Second Derivative Test if

$f(x) = x^2 e^{-2x^2}$.

$f'(x) = 0$ when $x = -0.7071, 0, 0.7071$

$f''(-0.7071) = -1.4715$ ∩ rel max

$f''(0) = 2$ ∪ rel min

$f''(0.7071) = -1.4715$ ∩ rel max

f has a relative min at $(0, 0)$ $\quad f(0) = 0$

f has a relative max $(-0.7071, 0.1839)$ $\quad f(-0.7071)$

$(0.7071, 0.1839)$ $\quad = f(0.7071)$

0.1839

Now we'll return to optimization problems, where we will work some problems where the function is not given.   These problems involve several steps:

1. The first step in each problem is to determine what you are trying to find. What is the objective of the problem? Are you maximizing something or minimizing something?

2. Next, write a function in one variable. In some situations, you will have to get two functions to work together to get the function written.   If possible, draw a picture of the situation. Choose variables for the values discussed and put them on your picture. Determine if there are any formulas you need to use, such as area or volume formulas. If you have a right triangle in your picture, decide if the Pythagorean Theorem will help.

3. Once you have a function, find any critical numbers. Since you are finding quantities such as dimensions, you'll only need to find positive critical numbers.

4. Next, you can use the Second Derivative Rest to verify that you have found a maximum (or a minimum, depending on your problem).

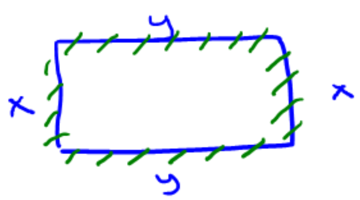
Remember that if the second derivative, evaluated at a number is positive, the function is concave upward at that point. And if the number is a critical number, you've shown that the function is concave upward and you have a minimum.

If the second derivative, evaluated at a number is negative, the function is concave downward at that point. If the number is a critical number, you've shown that the function is concave downward and you have a maximum.

5. Finally, make sure you answer the question(s) posed in the problem.

Example 3:   A man would like to have a rectangular shaped garden in his back yard. He has 100 feet of fencing to use to fence in the garden. Find the dimensions for the largest possible garden he can make if he uses all of the fencing.

area
maximize



objective:
maximize area

$A = xy$

we need a function
with just 1 variables

$$\boxed{A(x) = x(50-x)}$$

using GGB   c.n. $x = 25$
$f'(x)$
Root[f']

$F = 2x + 2y$

$100 = 2x + 2y$
available fencing

solve for y

$100 = 2x + 2y$
$-2x \quad -2x$
$\dfrac{100 - 2x}{2} = \dfrac{2y}{2}$

$50 - x = y$

Use Second Deriv. Test. Max? Min?

$A''(25) = -2$ $\cap$ max

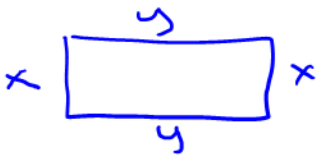
$x = 25$

$y = 50 - 25$

$y = 25$

$\boxed{\text{Square: } 25' \text{ by } 25'}$

Example 4: A man would like to have a rectangular shaped garden in his back yard. He wants the garden to have an area of 500 square feet. Find the dimensions for the garden that will require the least amount of fencing.



$A = 500 \text{ ft}^2$

$xy = 500$

solve for y

$\frac{xy}{x} = \frac{500}{x}$

$y = \frac{500}{x}$

objective:

minimize fencing

$F = 2x + 2y$

$\boxed{F(x) = 2x + 2\left(\frac{500}{x}\right)}$

use ggb; graph $F'(x)$

Roots[F', 10, 40]

c.n. $x = 22.3607$

2^{nd} deriv. test $F''(22.3607) = 0.1789$

rel. min

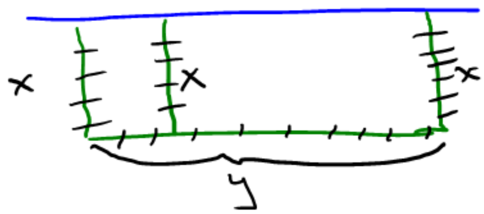
answer question

$x = 22.3607$

$y = \frac{500}{x} \rightarrow \frac{500}{22.3607} = 22.3607$

square $\boxed{22.3607 \text{ ft by } 22.3607 \text{ ft.}}$

Example 5: Suppose you wish to fence in a rectangular-shaped pasture that lies along the straight edge of a river. You will divide the pasture into two parts by means of a fence that runs perpendicular to the river and parallel to two of the sides of the pasture. You have 1500 meters of fencing to use, and you wish to fence in the maximum possible area. The side along the river will not be fenced. Determine the dimensions of the pasture that will provide the maximum area. What is that area?



$F = 3x + y$

$1500 = 3x + y$

$-3x \quad -3x$

solve for y

$1500 - 3x = y$

objective: maximize area

$A = xy$

$\boxed{A(x) = x(1500 - 3x)}$

use GGB to find critical number

c.n. $x = 250$

Test using 2nd deriv. test

$A''(250) = -6$ $\quad\cap\quad$ rel max

Dimensions

$x = 250$ $\qquad y = 1500 - 3x$

$y = 1500 - 3(250) = 750$

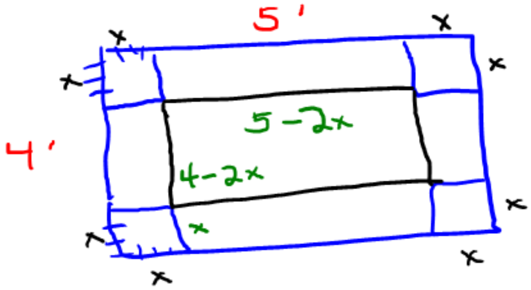
$\boxed{250\text{ m by } 750\text{ m}}$

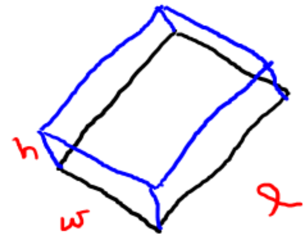
Area: $250 \times 750 = 187500$

$\boxed{187500\text{ m}^2}$

different from the others !

Example 6: If you cut away equal squares from all four corners of a piece of cardboard and fold up the sides, you will make a box with no top. Suppose you start with a piece of cardboard the measures 4 feet by 5 feet. Find the dimensions of the box that will give a maximum volume.





$\boxed{V(x) = x(5-2x)(4-2x)}$

objective: maximize volume

$V = lwh$

use GGB to find c.n.

$x = 0.7362$
$x = 2.2638$

2nd deriv test: $V''(0.7362) = -18.3303$ ∩ max

~~$V''(2.2638) = 18.3303$~~ ∪ min

answer

$x = 0.7362$

$5 - 2x = 3.5275$

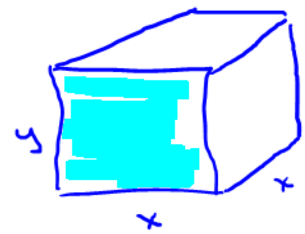
$4 - 2x = 2.5275$

$\boxed{3.5275 \text{ ft by } 2.5275 \text{ ft by } 0.7362 \text{ ft}}$

Note: "Open box" means the box has no top. "Closed box" means the box has a top.

Example 7: Suppose you want to minimize the amount of material that is needed by construct a closed box. The box will have a square base, and must have a volume of 100 cubic inches. What are the dimensions of the box that will use the smallest amount of material to construct?

objective: minimize surface area



Write formula for surface area

area of bottom +
area of top +
area of 4 sides

$V = lwh$

$V = x^2 y$

$100 = x^2 y$

solve for y

$\frac{100}{x^2} = y$

$SA = x^2 + x^2 + 4xy$

bottom top sides

$$\boxed{A(x) = x^2 + x^2 + 4x\left(\frac{100}{x^2}\right)}$$

$$\boxed{A(x) = 2x^2 + \frac{400}{x}}$$

use GGB to find critical #

$x = 4.6416$

2nd deriv. test

$A''(4.6416) = 12$ ∪

min

$x = 4.6416$

$y = \frac{100}{(4.6416)^2} = 4.6416$

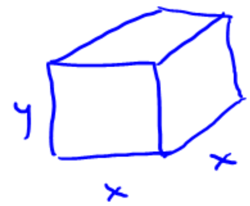
4.6416 in by 4.6416 in by 4.6416 in

(cube)

very similar to #7

Example 8: Suppose you want to minimize the cost of constructing a package. The package will have a square base and a top, and must have a volume of 250 cubic inches. The cost of the material used to construct the bottom of the box is 85 cents per square inch. The cost of the material used to construct the top of the box is 75 cents per square inch. The cost of the material used to construct the sides of the box is 55 cents per square inch. What are the dimensions of the box that will minimize the cost of the box?

objective : minimize cost



$SA = x^2 + x^2 + 4xy$

$V = x^2 y$

$250 = x^2 y$

$\frac{250}{x^2} = y$

$SA = x^2 + x^2 + 4x\left(\frac{250}{x^2}\right)$

now turn it into a cost function

$C(x) = 85x^2 + 75x^2 + 55 \cdot 4x\left(\frac{250}{x^2}\right)$

c.n. $x = 5.56$

2nd deriv. test $C''(5.56) = 960$ $\cup$ min

$x = 5.56$

$y = \frac{250}{5.56^2} = 8.0871$

$\boxed{\text{dimensions } 5.56 \text{ in by } 5.56 \text{ in by } 8.0871 \text{ in.}}$

A company's profits can be modeled by the function P(x)=-0.001x^3 -0.01x^2+1.08x+123.8 where P(x) is given in thousands of US dollars and 0 < x < 40 and x is given in thousands of units. What is the maximum profit that the company can expect? How many units must they sell to achieve that profit?

enter P(x) into GGB

Find critical numbers

c.n. ~~$x = -22.5976$~~

$x = 15.9309$

2nd deriv test: $P''(15.9309) = -0.1156$ max

$P(15.9309) = 134.4243$

$$\boxed{\begin{array}{l}\text{Max profit } \$\ 134\,424^{30} \\ \text{when } 15931 \text{ units sold}\end{array}}$$

Example 9: Suppose a travel agency charges $350 per person for a charter flight if 200 people purchase seats on the plane. As an incentive, for each person over 200 that books the flight, the agency will deduct $1 from everyone's ticket price. How many passengers will result in maximum revenue for the travel agency? What is that revenue? What would be the fare per person?

(# that book − 200)

let x = # of people above 200 that book the flight

$R(x)$ = price × # of items sold

price = $350 - 1x$

# of seats = $200 + x$

$R(x) = (350 - x)(200 + x)$

critical number = 75

$R''(75) = -2$ ∩ max

# of passengers = 200 + 75 = 275

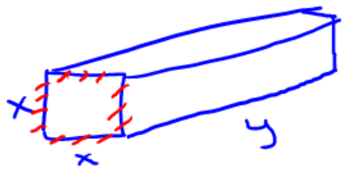
$R(75) = 75\,625$

Max revenue is $75625.00

airfare per person = 350 − 75 = 275

$\boxed{\$275}$

Example 10: Postal regulations state that the girth plus length of a package must be no more than 104 inches if it is to be mailed through the US Postal Service. You are assigned to design a package with a square base that will contain the maximum volume that can be shipped under these requirements. What should be the dimensions of the package? (Note: girth of a package is the perimeter of its base.)



objective: maximize volume

$V = x^2 y$

girth = $4x$

girth + length = $4x + y$

$104 = 4x + y$

$104 - 4x = y$

$V(x) = x^2(104 - 4x)$

c.n. $x = 17.3333$

2nd deriv. test

$V''(17.3333) = -208$ $\cap$ max

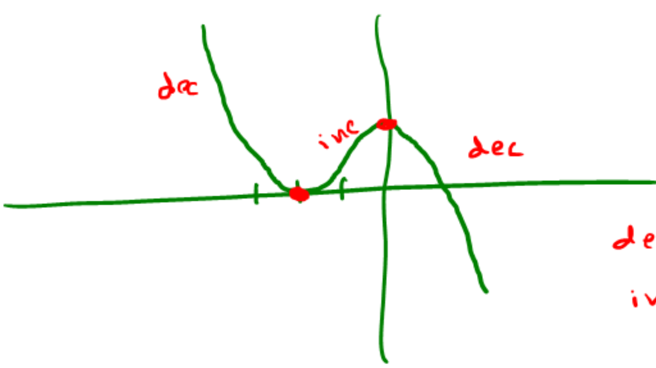
$x = 17.3333$

$y = 104 - 4x$

$= 104 - 4(17.3333)$

$= 34.6668$

$\boxed{17.3333 \text{ in by } 17.3333 \text{ in by } 34.6668 \text{ in}}$



dec $(-\infty, -2) \cup (0, \infty)$

inc $(-2, 0)$