domain | inc/dec | conc ↑ / conc ↓
zeros, y int | relative extrema | inflection pts
asymptotes | |
function | f ' | f ''

Math 1314

Lesson 13: Analyzing Other Types of Functions

If the function you need to analyze is something other than a polynomial function, you will have some other types of information to find and some analysis techniques will be slightly different.

We can find critical numbers by looking at a graph of the derivative of a function. When working with the critical numbers of a polynomial, we defined critical numbers to be all values of *x* that are in the domain of a function where $f'(x)=0$. When working with other types of functions, *critical numbers will also include any numbers in the domain of the function for which the derivative is not defined.* We'll need to consider values of x that are in the domain of the function and for which the graph of the derivative has asymptotes, sharp turns and vertical tangent lines.

The **critical numbers** of a function are numbers in the domain of the function where $f'(x)=0$ or where $f'(x)$ is undefined.

Example 1: The graph shown below is the derivative of a function *f*. The domain of the function is $(-\infty,-2)\cup(-2,\infty)$. Find any critical numbers.

where is f'(x) = 0?

x = 2.8

where is f'(x) undefined

x = -2

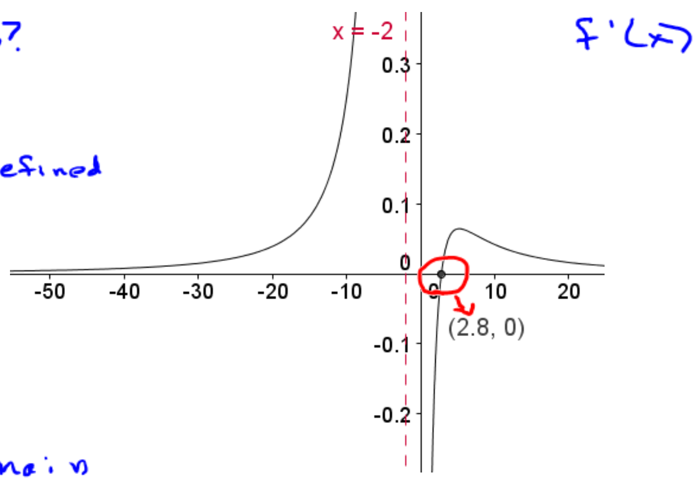
(V.A. @ x = -2)

are they in the domain?

2.8 is in domain

-2 is not

critical number is 2.8



$f'(x) = 0$

① find $f'(x)$
② set $= 0$, solve for x

POPPER 9, question 1: $f'(x) = 6x - 18$

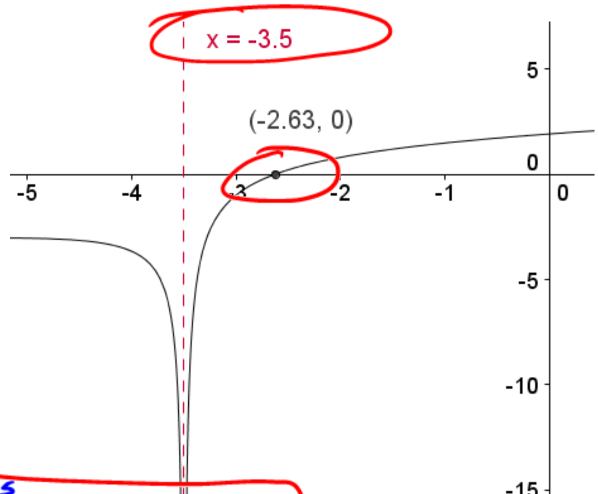
$6x - 18 = 0$
$6x = 18$
$\boxed{x = 3}$

Suppose $f(x) = 3x^2 - 18x + 5$. What are the critical numbers?

A. 5 B. 0, 6 C. 3 D. 4 E. 0

Example 2: The graph shown below is the derivative of a function f. The domain of the function is $(-\infty, \infty)$. Find any critical numbers.

where is $f'(x) = 0$
$x = -2.63$
where is $f'(x)$ undefined?
$x = -3.5$
are these in the domain?
yes



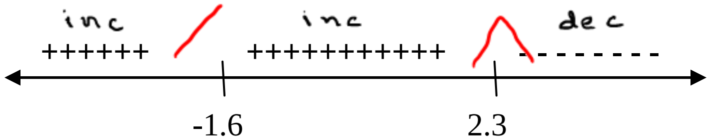
critical numbers
$x = -3.5$, $x = -2.63$

You must use caution when the graph of the derivative shows an asymptote, a vertical tangent line or a sharp turn in the graph of the function. If this point is in the domain of the function, *it is also a critical number and must be used in the analysis of the function*.

After this adjustment in finding critical numbers, the analysis for finding increasing/decreasing intervals and relative extrema is the same as what we used in analyzing polynomial functions. Note that any vertical asymptotes must be used in subdividing the number line along with all critical numbers, but a vertical asymptote cannot generate a relative extremum. Note also that GGB commands are different. To find zeros of the derivative, you must use the command Roots[<Function>, <Start x-Value>, <End x-Value>], as these functions are not polynomials.

POPPER 9, question 2:

This is the first derivative number line you created while analyzing the sign of the first derivative. Which of the statements given below is true?



~~A.~~ There is a relative minimum when x = 2.3.
~~B.~~ The function is decreasing on two intervals.
~~C.~~ There is a relative maximum when x = -1.6.
~~D.~~ There is no extremum when x = 2.3.
E. There is no extremum when x = -1.6.

Example 3: Find any critical numbers. Then find intervals on which the function is increasing and intervals on which the function is decreasing. Then find any relative extrema.

$$f(x) = \frac{2x^2 - 1}{x+1}$$

domain $x \neq -1$
$(-\infty, -1) \cup (-1, \infty)$

$f'(x) = 0$ when $x = -1.7071$, $x = -0.2929$
$f'(x)$ is undefined when $x = -1$ (V.A)

$\boxed{\text{critical numbers } x = -1.7071, -0.2929}$

first deriv. line test

$\boxed{f \text{ is inc } (-\infty, -1.7071) \cup (-0.2929, \infty) \quad f \text{ is dec. on } (-1.7071, -1) \cup (-1, -0.2929)}$

$\boxed{\text{rel. max at } (-1.7071, -6.8284) \quad \text{rel min at } (-0.2929, -1.1716)}$

Example 4: Find any critical numbers. Then find intervals on which the function is increasing and intervals on which the function is decreasing. Then find any relative extrema.

$$f(x) = (x+3)^{\frac{1}{3}} = \sqrt[3]{x+3}$$

domain $(-\infty, \infty)$

$f'(x) \neq 0$ for any x

$f'(x)$ is undefined when $x = -3$

$\boxed{\text{critical number } x = -3}$

first deriv. line

$\boxed{f \text{ is inc. on } (-\infty, -3) \cup (-3, \infty)}$ (or $(-\infty, \infty)$)

(f is never decreasing)

$\boxed{\text{no relative extrema}}$

Example 5: Find any critical numbers. Then find intervals on which the function is increasing and intervals on which the function is decreasing. Then find any relative extrema.

domain $(-\infty, \infty)$

$$f(x) = x^2 e^{-x}$$

$f'(x) = 0$ when $x = 0$, $x = 2$

$f'(x)$ is never undefined

$\boxed{\text{critical numbers } x = 0, 2}$

f' line

$$\underset{}{- - - - -}\ \overset{\vee}{\underset{0}{|}}\ + + + +\ \overset{\wedge}{\underset{2}{|}}\ - - - - -$$

f is dec. on $(-\infty, 0) \cup (2, \infty)$
f is inc. on $(0, 2)$

relative min at $(0, 0)$
relative max at $(2, 0.5413)$

POPPER 9, question 3

True or false: A function can only have a relative extremum at a critical number.

A. True

B. False

The technique you will use to find concavity intervals and inflections points of a general function is very similar to the technique you used to find these features of a polynomial function.

First, find all values in the domain of the function for which either $f''(x)=0$ or $f''(x)$ is undefined. Note also the location of any vertical asymptotes of the function.

Next, draw a number line and subdivide it at all values you identified in the previous step. Note the sign of the second derivative in each interval.

Finally, read off the results. If the second derivative is positive on an interval, the function is concave upward in that interval. If the second derivative is negative on an interval, then the function is concave downward on that interval. If the second derivative changes signs at a value c that is not a vertical asymptote, then the function has an inflection point when $x=c$. Remember that a function cannot have an inflection point where the function has a vertical asymptote.

Example 6: Use the graph of the second derivative of $f(x)=\dfrac{2x^2-1}{x+1}$ to describe the concavity of the function. Then find any inflection points.

domain $(-\infty,-1)\cup(-1,\infty)$

$f''(x)\neq 0$ for any x

$f''(x)$ is undefined at $x=-1$ (not in domain)

f'' line $\quad - - - \cap - - \;$ V.A $\; + + \cup + + +$; -1

$\boxed{f \text{ is conc} \downarrow (-\infty,-1) \qquad f \text{ is conc} \uparrow (1,\infty)}$

$\boxed{f \text{ has no inflection points}}$

Example 7: Use the graph of the second derivative of $f(x) = (x+3)^{\frac{1}{3}}$ to describe the concavity of the function. Then find any inflection points.

domain $(-\infty, \infty)$

$f''(x) \neq 0$ for any x

$f''(x)$ is undefined when $x = -3$ (in the domain)

f'' $\cup$ $+ + + + + + +$ | $-$ $-$ $-$ $-$ $-$ $\cap$ (-3)

$\boxed{f \text{ is conc} \uparrow \text{ on } (-\infty, -3) \quad f \text{ is conc} \downarrow \text{ on } (-3, \infty)}$

$\boxed{f \text{ has an inflection point at } (-3, 0)}$

Example 8: Use the graph of the second derivative of $f(x) = x^2 e^{-x}$ to describe the concavity of the function. Then find any inflection points.

$f''(x) = 0$ at $x = 0.5858, 3.4142$

$f''(x)$ is never undefined

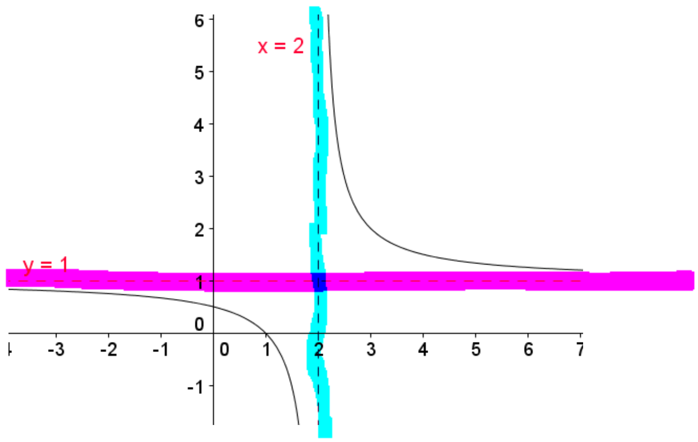
f'' line $\cup$ $+ + + +$ | $-$ $-$ $\cap$ $-$ $-$ $-$ | $+ + + + + + +$ $\cup$ (0.5858, 3.4142)

$\boxed{\begin{array}{l} f \text{ is conc} \uparrow \ (-\infty, 0.5858) \cup (3.4142, \infty) \\ f \text{ is conc} \downarrow \ (0.5858, 3.4142) \end{array}}$

$\boxed{\begin{array}{l} f \text{ has inflection points at } (0.5858, 0.191) \\ \qquad (3.4142, 0.3835) \end{array}}$

Asymptotes

We also want to identify any vertical or horizontal asymptotes of the graph of a function. A vertical asymptote is a vertical line $x = a$ that the graph approaches as values for x get closer and closer to a. A horizontal asymptote is a horizontal line $y = b$ that the graph of a function approaches as values for x increase without bound. The graph of the function shown below has a vertical asymptote at $x = 2$ and a horizontal asymptote at $y = 1$.



If you are given the graph of the function, you can usually just find asymptotes visually by looking at the graph of the function.

When given a function (and not a graph), you can use these rules for finding asymptotes.

Polynomial functions have no asymptotes.

Rational functions may have vertical asymptote(s), horizontal asymptote(s), both or neither. To determine asymptotes of rational functions, use these rules:

- To find a vertical asymptote of a function, reduce the function to lowest terms; then set the denominator equal to zero and solve for x.
- To find a horizontal asymptote, find $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$. To do this, compare the degree of the numerator and the degree of the denominator.
  - If $\deg(num) < \deg(den)$, the horizontal asymptote is at $y = 0$.
  - If $\deg(num) = \deg(den)$, where p is the leading coefficient of the numerator and q is the leading coefficient of the denominator, then the horizontal asymptote is $y = \frac{p}{q}$. Reduce the fraction to lowest terms, if possible.
  - If $\deg(num) > \deg(den)$, then the graph of the function does not have a horizontal asymptote.

BOBO
BOTN
EATS
DC

An exponential function may have a horizontal asymptote. An exponential function of the form $f(x) = c + a \cdot b^x$ or $g(x) = c + a \cdot e^{bx}$ will have a horizontal asymptote at $y = c$.
A logarithmic function may have a vertical asymptote. A logarithmic function of the form $f(x) = c + a \cdot \log_b(x - d)$ or $g(x) = c + a \cdot \ln(x - d)$ will have a vertical asymptote at $x = d$.

There is also an Asymptote command in GGB.

Example 9: Find any asymptotes: $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2} = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-1)}$

hole @ $x - 2 = 0$
$x = 2$

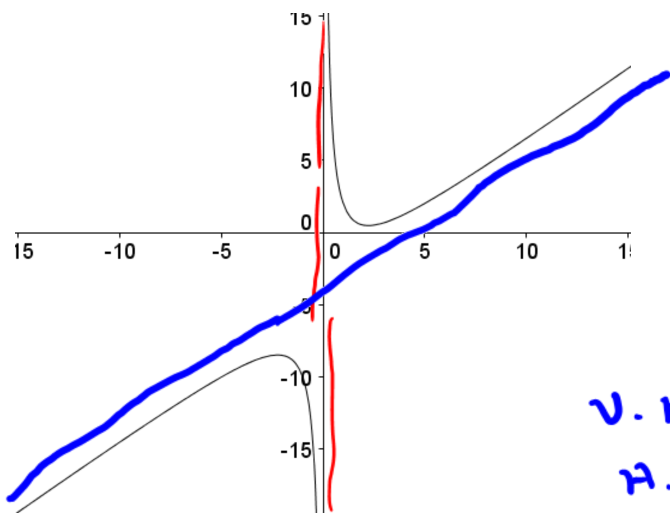
$g(x) = \dfrac{x+2}{x-1}$

V.A. $x - 1 = 0$
$x = 1$

H.A. $y = 1$

POPPER 9, question 4:

Look at the graph that is shown below, then select the true statement from the list below the graph.



V.A. ✓
H.A. ×

A. The function has a horizontal asymptote but not a vertical asymptote.
B. The function has both a horizontal asymptote and a vertical asymptote.
C. The function has a vertical asymptote but not a horizontal asymptote.
D. The function has neither a vertical asymptote nor a horizontal asymptote.

Now we'll find all features of two functions.

The analysis will include:

Domain
Coordinates of any zeros
Equations of any asymptotes
Interval(s) on which the function is increasing; interval(s) on which the function is decreasing
Coordinates of any relative extrema
Interval(s) on which the function is concave upward; interval(s) on which the function is concave downward
Coordinates of any inflection points

Analysis summary

from $f(x)$

① domain

② x int — Root [polynomial]

Roots [function, start, end]

③ y int — $f(0)$

④ asymptotes — $x = a$ V.A., $y = b$ H.A.

Asymptote [function]

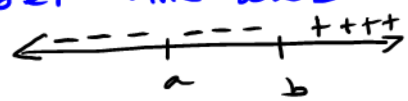
check it! May not be accurate

$y = 2x - 5$ (slant asympt) ignore

from $f'(x)$

⑤ critical numbers

⑥ first derivative number line which gives



ⓐ inc/dec intervals

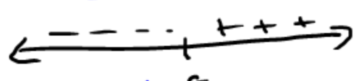
ⓑ relative extrema

for y values find $f(a)$

from $f''(x)$

⑦ where $f''(x) = 0$ or is undefined

⑧ 2nd derivative number line which gives



ⓐ conc↑ / conc↓ intervals

ⓑ inflection points

for y values find $f(c)$

Example 14: Analyze the function $f(x) = x^2 e^{-0.5x} - 1$.

f

domain $(-\infty, \infty)$

zeros $x = -0.8156,\ 1.4296,\ 8.6132$

H.A. $y = -1$

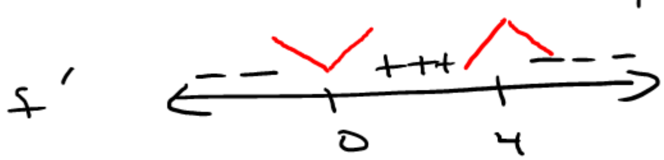
y int $(0, -1)$

f'

$f'(x) = 0$ when $x = 0$, $x = 4$

$f'(x)$ is always defined

critical numbers $x = 0, 4$



f is inc on $(0, 4)$

f is dec on $(-\infty, 0) \cup (4, \infty)$

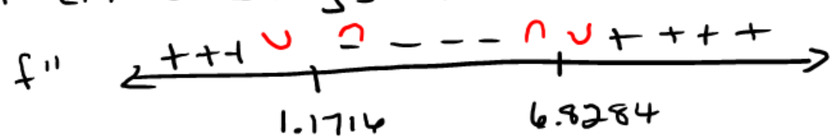
f has a relative minimum at $(0, -1)$

f has a relative maximum at $(4, 1.1654)$

f''

$f''(x) = 0$ when $x = 1.1716$, $x = 6.8284$

$f''(x)$ is always defined



f is conc $\uparrow$ on $(-\infty, 1.1716) \cup (6.8284, \infty)$

f is conc $\downarrow$ on $(1.1716, 6.8284)$

inflection points $(1.1716, -0.2359)$

$(6.8284, 0.5341)$

open now

POPPER 9, question 6: B

Example 15: Analyze the function $f(x) = \dfrac{x^3 - 4x^2 - 4}{x}$.

$\underline{f}$

domain $x \neq 0$

$(-\infty, 0) \cup (0, \infty)$

x int $(4.2242, 0)$

y int none $(f(0) = -\infty) \rightarrow$ no y int

asymptotes V.A. $x = 0$

$\underline{f'}$

$f'(x) = 0$ when $x = -0.8393$

$f'(x)$ is undefined when $x = 0$ (not in domain)

critical numbers $x = -0.8393$

f': $- - -$ | $+ + + + +$ V.A $+ + + +$; marks at -0.8393 and 0

f is inc on $(-0.8393, 0) \cup (0, \infty)$

f is dec on $(-\infty, -0.8393)$

f has a relative minimum at $(-0.8393, 8.8275)$

f has no relative maxima

f''

$f''(x) = 0$ when $x = 1.5874$

$f''(x)$ is undefind when $x = 0$ (not in domain)

f'': $+ + + + +$ V.A $- - - \cap$ | $\cup$ $+ + + + + +$; marks at 0 and 1.5874

f is conc $\uparrow$ on $(-\infty, 0) \cup (1.5874, \infty)$

f is conc $\downarrow$ on $(0, 1.5874)$

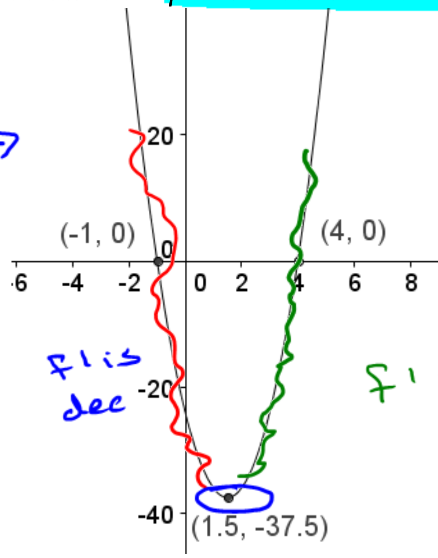
inflection point $(1.5874, -6.3496)$

If all you have to work with is the graph of the first derivative of a function, you can still find out a lot of information about the function.

Example 16: The graph given below is the *first derivative* of a function, *f*.

f' +++ ---- ++++
-1 4

f'' ----- ++++
1.5



f' is inc

Find the interval(s) on which the function is increasing, the interval(s) on which the function is decreasing.

f is inc on $(-\infty,-1) \cup (4,\infty)$

Find the *x* coordinate of each relative extremum (and state whether it is a relative maximum or a relative minimum).

f has rel max at $x=-1$

f has rel min at $x=4$

Find the interval(s) on which the function is concave upward, the interval(s) on which the function is concave downward.

where is f' increasing

where is f' decreasing

f is conc ↑ on $(1.5,\infty)$

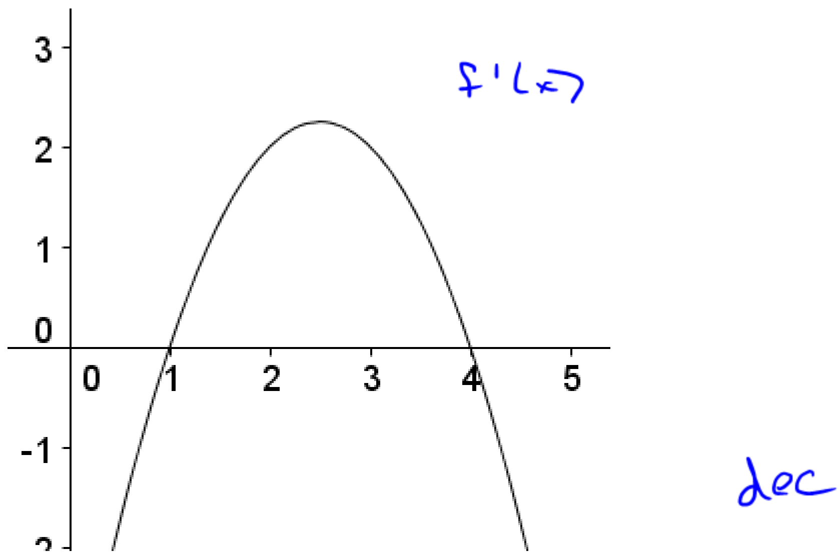
f is conc ↓ on $(-\infty,1.5)$

Find the *x* coordinate of any inflection points.

$x=1.5$

POPPER 9, question 5:

The graph given below is the graph of the first derivative of a function. On what interval(s) is the *function* decreasing?



A. $(1, 4)$ B. $(0,1)\cup(4,5)$ C. $(-\infty, 1)\cup(4,\infty)$