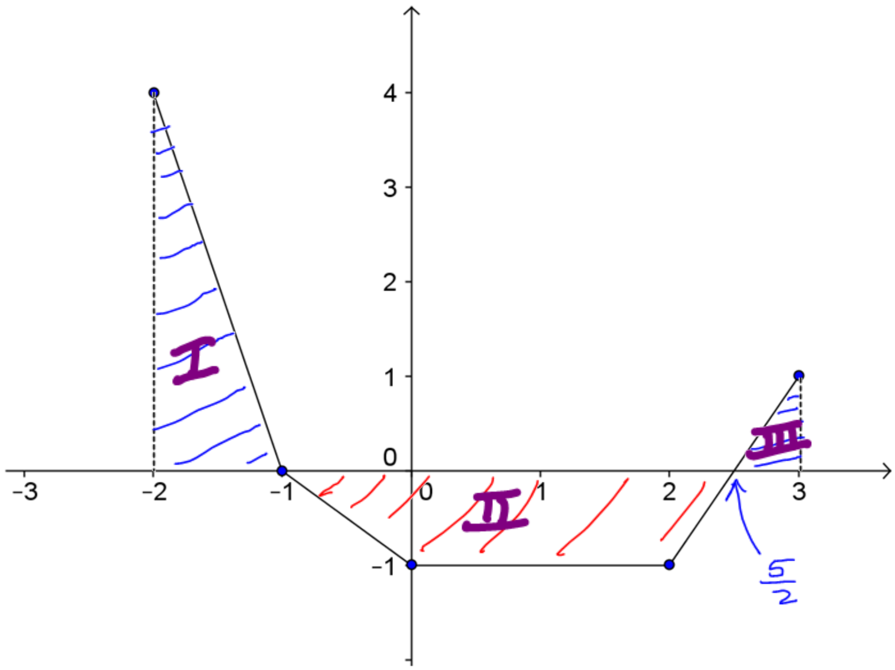
1. Give the average value of $f(x) = \cos(2x)$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{4}\right]$.

$$= \frac{1}{\frac{\pi}{4} - \frac{-\pi}{2}} \int_{-\pi/2}^{\pi/4} \cos(2x)\,dx =$$

$$= \frac{4}{3\pi} \cdot \frac{1}{2} \sin(2x) \Big|_{-\pi/2}^{\pi/4} = \frac{2}{3\pi}\left[1 - 0\right] = \frac{2}{3\pi}$$

2. The graph of $f(x)$ is shown below.



a. Give the area of the region bounded between the graph of $f(x)$ and the x-axis on the interval $[-2, 3]$.

b. $\int_{-2}^{3} f(x)dx =$

(a) $\text{Area}(\text{I}) + \text{Area}(\text{II}) + \text{Area}(\text{III})$

$= \frac{1}{2}\cdot 4 \cdot 1 + \frac{1}{2}\left(\frac{7}{2} + 2\right)\cdot 1 + \frac{1}{2}\cdot\frac{1}{2}\cdot 1$

$= 2 + \frac{11}{4} + \frac{1}{4} = 5$

(b) $\int_{-2}^{3} f(x)\,dx = \text{Area}(\text{I}) - \text{Area}(\text{II}) + \text{Area}(\text{III})$

$= 2 - \frac{11}{4} + \frac{1}{4} = 2 - \frac{5}{2} = -\frac{1}{2}$

3. Give the average value of $f(x) = x^2 - 2x + 4$ on the interval [–1,2], and verify the conclusion of the mean value theorem for integrals for this function on this interval.

$$\text{Average Value} = \frac{1}{2--1}\int_{-1}^{2}(x^2-2x+4)\,dx$$

$$= \frac{1}{3}\left(\frac{1}{3}x^3 - x^2 + 4x\right)\Big|_{-1}^{2}$$

$$= \frac{1}{3}\cdot 12 = 4$$

Note that $f(x)$ is a polynomial.

$\therefore$ $f(x)$ is continuous on $[-1,2]$.

We verify the conclusion of the MVThm for integrals by solving

$f(c) = 4$ $\leftarrow$ the average value

for a value c in $(-1,2)$.

$$c^2 - 2c + 4 = 4$$

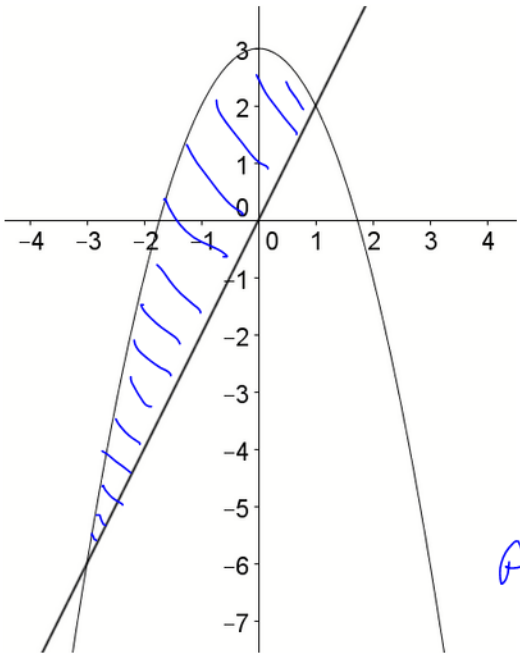
$$c^2 - 2c = 0$$

$$c(c-2) = 0$$

$$\boxed{c=0} \quad \text{or} \quad \cancel{c = 2}$$

$c = 0$ is in $(-1,2)$ and verifies the conclusion of the MVThm for integrals.

4. Sketch the region bounded between the graphs of $f(x)=3-x^2$ and $g(x)=2x$. Then find the area of the region.



$f(x)=g(x)$ when

$3-x^2=2x$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3$ or $x=1$.

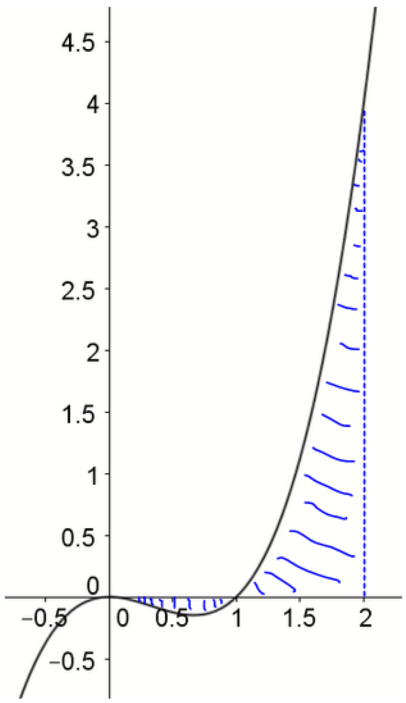
Also, $f(x)\ge g(x)$ on $[-3,1]$.

$$\text{Area}=\int_{-3}^{1}(f(x)-g(x))\,dx$$

$$=\int_{-3}^{1}(3-x^2-2x)\,dx$$

$$=\frac{32}{3}$$

5. Find the **area** bounded by the graph of $f(x) = x^3 - x^2$ and the x-axis on the interval [0,2].



$f(x) = 0 \iff$

$x^3 - x^2 = 0$

$x^2(x-1) = 0$

$x = 0$ or $x = 1$

Also, $f(x) \leq 0$ on $[0,1]$

and $f(x) \geq 0$ on $[1,2]$.

$\therefore$ Area $= -\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$= -\int_0^1 (x^3 - x^2)\,dx + \int_1^2 (x^3 - x^2)\,dx$

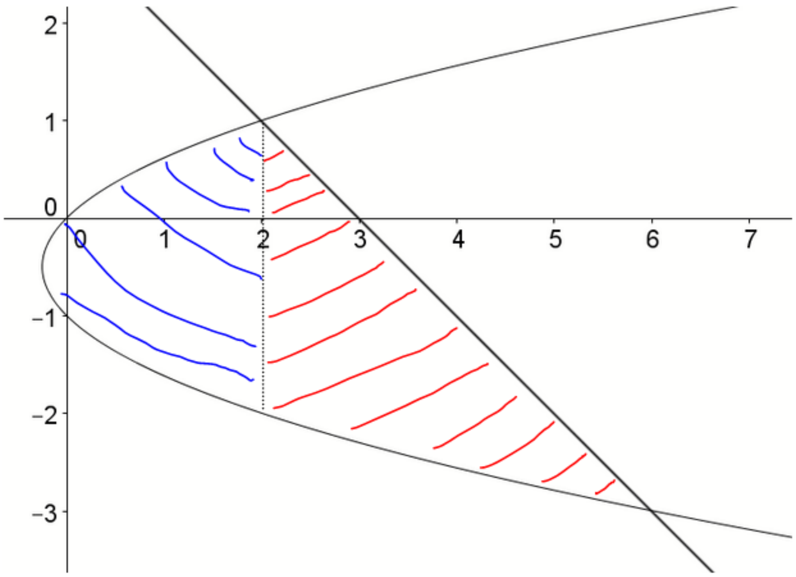
$= -\left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_1^2$

$= \frac{1}{12} + \left(\left(4 - \frac{8}{3}\right) - \left(-\frac{1}{12}\right)\right)$

$= \frac{1}{6} + 4 - \frac{8}{3} = \frac{1 + 24 - 16}{6}$

$= 3/2$

6. Sketch the region bounded by the curves $x+y=3$ and $x=y^2+y$. Then give a formula for the area of the region involving integral(s) in x. Repeat the process with integral(s) in y. Finally, find the area of the region.



Integrals in x:

$x+y=3 \Leftrightarrow y=3-x$ $(x=3-y)$

$x=y^2+y \Leftrightarrow y^2+y-x=0$

$$y=\frac{-1\pm\sqrt{1+4x}}{2}$$

$$y=-\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4x}$$

It is easier to find the intersection in y, and then use it to determine x.

$3-y=y^2+y \Rightarrow y^2+2y-3=0 \Rightarrow (y+3)(y-1)=0$

$y=-3$ or $y=1$

$\Updownarrow$ $(x=3-y)$ $\Updownarrow$

$x=6$ $x=2$

Also, the vertex of $x=y^2+y$ is at $y=-\frac{1}{2}$

$\Rightarrow x=-\frac{1}{4}$

Using integrals in x:

$$\text{Area}=\int_{-\frac{1}{4}}^{2}\left(\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4x}\right)-\left(-\frac{1}{2}-\frac{1}{2}\sqrt{1+4x}\right)\right)dx+\int_{2}^{6}\left((3-y)-\left(-\frac{1}{2}-\frac{1}{2}\sqrt{1+4x}\right)\right)dx$$

$$=\cdots=\frac{32}{3}$$

Integrals in y:

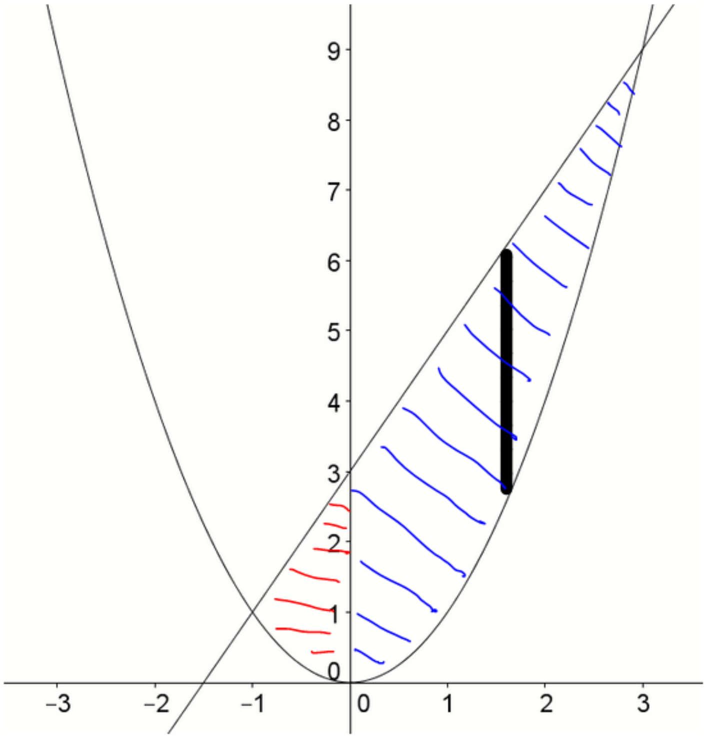
$$\text{Area} = \int_{-3}^{1} \left((3-y) - (y^2+y) \right) dy$$

$$= \int_{-3}^{1} (3 - 2y - y^2)\, dy$$

$$= \left(3y - y^2 - \frac{1}{3} y^3 \right) \Big|_{-3}^{1}$$

$$= \dots = \frac{32}{3}$$

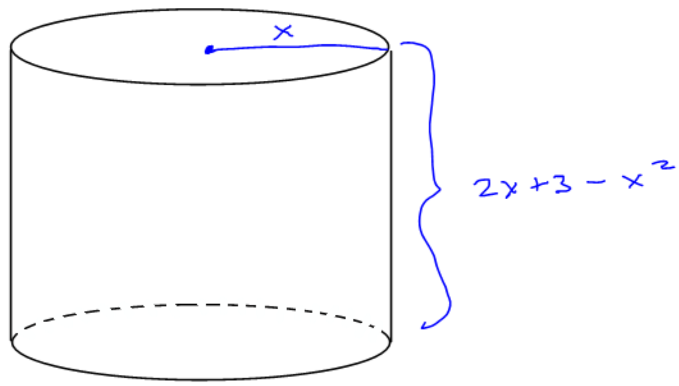
7. Sketch the region bounded between $f(x) = 2x + 3$ and $g(x) = x^2$. Rotate this region around the y-axis to generate a solid, and then find the volume of the solid.



Note: From symmetry, the solid generated when this region is rotated around the y axis is the same as the solid generated when the region in the first quadrant is rotated around the y axis.

Consequently, we neglect the RED region.

The vertical line drawn on the left rotates around the y-axis to form a cylindrical shell, as shown below.



$$2x+3 = x^2 \iff x^2 - 2x - 3 = 0$$

$$\iff (x+1)(x-3) = 0$$

$$\iff x = -1 \text{ or } x = 3$$

$$\text{Volume} = \int_0^3 2\pi \cdot x \cdot (2x + 3 - x^2)\, dx$$

$$= \cdots = \frac{45\pi}{2}$$

8. Sketch the region **in the first quadrant** bounded between $f(x) = 2x + 3$ and $g(x) = x^2$.

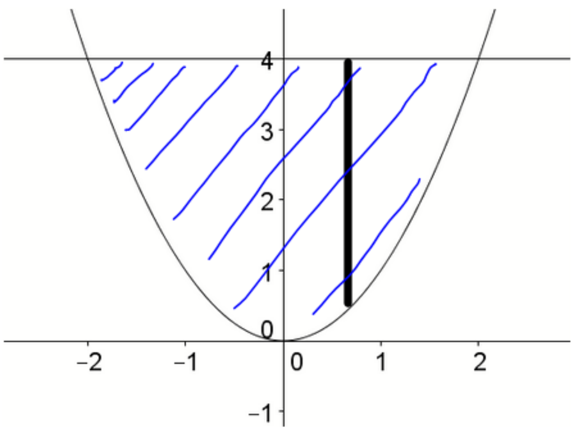
Rotate this region around the *y*-axis to generate a solid, and then find the volume of the solid.

See problem 7.

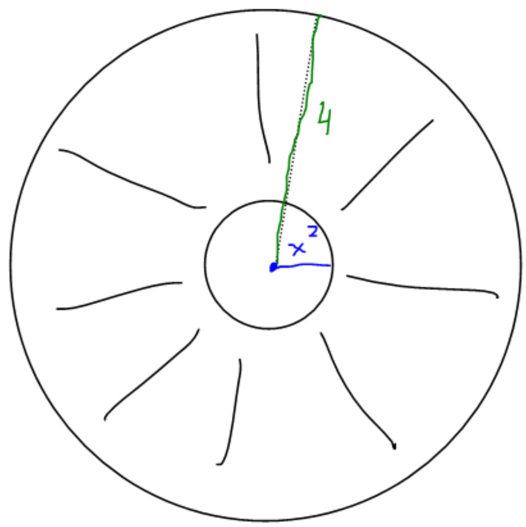
The result is the same.

9. Revolve the region bounded by the line $y = 4$ and the graph of $f(x) = x^2$ about the x-axis to generate a solid. Find the volume.

$4 = x^2$ iff $x = \pm 2$



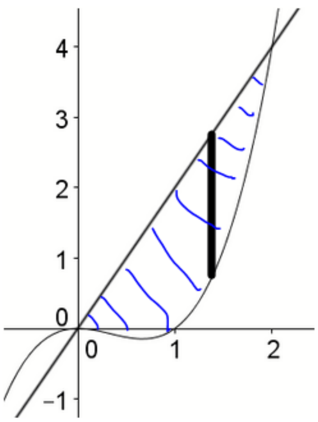
Revolving the vertical line around the x axis gives the washer shown below.



$$\text{Volume} = \int_{-2}^{2} \pi\left((4)^2 - (x^2)^2\right) dx$$

$$= \pi \int_{-2}^{2} (16 - x^4)\, dx = \cdots = \frac{256\pi}{5}$$

10. The region bounded between the graphs of $f(x) = x^3 - x^2$ and $g(x) = 2x$ on the interval [0,2] is rotated around the y-axis to generate a solid. Find the volume.



$f(x) = g(x) \iff x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$

$x(x-2)(x+1) = 0$

$x = 0,\ x = 2,\ \cancel{x = -1}$

Rotating the vertical line around the y axis generates a cylindrical shell with

height = $2x - (x^3 - x^2) = 2x - x^3 + x^2$

and **radius =** x

$$\text{Volume} = \int_0^2 2\pi \cdot x \cdot (2x - x^3 + x^2)\, dx = \cdots = \frac{88\pi}{15}$$