1. Riemann sums. Sketch the region associated with the approximation, and then given the requested value.
   a. Give the upper sum for \( f(x) = 4 - x^2 \) on the interval \([-1,1]\) with respect to the partition
      \[ P = \left\{ -1, -\frac{1}{2}, \frac{1}{2}, 1 \right\} \].
   b. Give the lower sum for \( f(x) = 4 - x^2 \) on the interval \([-1,1]\) with respect to the partition
      \[ P = \left\{ -1, -\frac{1}{2}, \frac{1}{2}, 1 \right\} \].
   c. Give the midpoint approximation for \( \int_{-1}^{1} (4 - x^2) \, dx \) with respect to the partition
      \[ P = \left\{ -1, -\frac{1}{2}, \frac{1}{2}, 1 \right\} \].
   d. Give the midpoint approximation for \( \int_{-1}^{1} (4 - x^2) \, dx \) using \( n = 4 \).
   e. Give the left hand endpoint approximation for \( \int_{-1}^{1} (4 - x^2) \, dx \) using \( n = 4 \).
   f. Give the right hand endpoint approximation for \( \int_{-1}^{1} (4 - x^2) \, dx \) using \( n = 4 \).
   g. Use the graph of \( f(x) \) below.

   ![Graph](image)

   Give the upper sum for \( f(x) \) on the interval \([-1,3]\) with respect to the partition
   \[ P = \left\{ -1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 3 \right\} \].
Riemann sums. Sketch the region associated with the approximation, and then given the requested value.

a. Give the upper sum for $f(x) = 4 - x^2$ on the interval [-1,1] with respect to the partition

$$P = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}.$$  

The upper sum for $f(x)$ with respect to $P$ is

$$\mathcal{U}_f(P) = f(-1) \cdot \frac{1}{2} + f(0) \cdot 1 + f(1) \cdot \frac{1}{2}$$

$$= 7.75$$

b. Give the lower sum for $f(x) = 4 - x^2$ on the interval [-1,1] with respect to the partition

$$P = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}.$$  

The lower sum for $f(x)$ with respect to $P$ is

$$\mathcal{L}_f(P) = f(-1) \cdot \frac{1}{2} + f(\frac{1}{2}) \cdot 1 + f(1) \cdot \frac{1}{2}$$

$$= 6.75$$
c. Give the midpoint approximation for $\int_{-1}^{1} (4 - x^2) \, dx$ with respect to the partition $P = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$.

\[
\approx f(-.75) \cdot \frac{1}{2} + f(0) \cdot 1 + f(.75) \cdot \frac{1}{2} = 7.4375
\]

Note: $\int_{-1}^{1} (4 - x^2) \, dx = \frac{22}{3} = 7.33\\$

So the midpoint approximation is more accurate than the approximations in (a) and (b).

However, the average of (a) and (b)
\[
\frac{1}{2} (7.75 + 6.75) = 7.25
\]

which is a decent approximation.

In general, the average of the upper and lower sums will give a reasonable approximation of the definite integral, as will the midpoint approximation. The upper and lower sums typically do not do as good a job of approximating the definite integral.
d. Give the midpoint approximation for \( \int_{-1}^{1} (4 - x^2) \, dx \) using \( n = 4 \).

\[
\approx \frac{1}{2} \left( f(-.75) + f(-.25) + f(.25) + f(.75) \right) \\
= 7.375
\]

---

e. Give the left hand endpoint approximation for \( \int_{-1}^{1} (4 - x^2) \, dx \) using \( n = 4 \).

\[
\approx \frac{1}{2} \left( f(-1) + f(-.5) + f(0) + f(.5) \right) \\
= 7.25
\]
f. Give the right hand endpoint approximation for \( \int_{-1}^{1} (4 - x^2) \, dx \) using \( n = 4 \).

\[ \approx \frac{1}{2} \left( f(-0.5) + f(0) + f(0.5) + f(1) \right) \]

\[ = 7.25 \]

g. Use the graph of \( f(x) \) below.

\[ y = x + 3 \]

\[ y = \frac{3}{2}x + 3 \]

\[ y = \frac{3}{2} \cdot x - y \]

Give the upper sum for \( f(x) \) on the interval \([-1, 3]\) with respect to the partition

\[ P = \left\{-1, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 3\right\}. \]

\[ \bigcup_{f} (P) \approx f(-\frac{1}{2}) \cdot \frac{1}{2} + f(0) \cdot 1 + f(\frac{1}{2}) \cdot 1 + f(\frac{3}{2}) \cdot 1 + f(3) \cdot \frac{1}{2} \]

\[ = \frac{5}{2} \cdot \frac{1}{2} + 3 \cdot 1 + 2 \cdot 25 \cdot 1 + 1 \cdot 1 + 2 \cdot \frac{1}{2} \]

\[ = 8.5 \]
2. Fundamental theorem of calculus.

a. \( \frac{d}{dx} \int_{\pi}^{x} \sin(t) \, dt = \sin(x) \)

b. \( \frac{d}{dx} \int_{3x}^{\pi} \sin(t^2) \, dt = -\sin((3x^2) \cdot 3) = -3\sin(9x^2) \)

c. \( \frac{d}{dx} \int_{3x}^{2x} \sin(t^2) \, dt = \sin((2x^2) \cdot 2) - \sin((3x^2) \cdot 3) = 2\sin(4x^2) - 3\sin(9x^2) \)

d. \( \frac{d}{dx} \int_{1-2x}^{2x} \sin(t^2) \, dt = -\sin((2x^2) \cdot 6x^2) - \sin((1-2x)^2) \cdot (-2) = 6x^2 \sin(4x^4) + 2\sin((1-2x)^2) \)
3. Basic Integration.

a. \[ \int_{0}^{7} x\sqrt{x^2 + 2} \, dx = \]

b. \[ \int_{0}^{\pi/4} \sec^2 (r) - 2\sqrt{r - 1} \, dr = \]

c. \[ \int_{0}^{1} \frac{\cos (x)}{2 + \sin (x)} \, dx = \]

d. \[ \int_{0}^{1} e^{-2x} \, dx = \]

e. \[ \int_{0}^{\pi/4} \frac{2}{1 + x^2} \, dx = \]

f. \[ \int_{0}^{\pi/4} \sec (x) \tan (x) \, dx = \]

g. \[ \int_{0}^{\pi/4} \frac{3t}{t^2 + 1} \, dt = \]

\[ \int_{2}^{7} x \sqrt{x^2 + 2} \, dx = \frac{1}{2} \int_{2}^{7} \sqrt{\frac{x^4 + 2}{u}} \cdot 2x \, dx \]

\[ u = x^2 + 2 \quad \Rightarrow \quad u = 51 \]

\[ x = 7 \quad \Rightarrow \quad u = 51 \]

\[ x = 2 \quad \Rightarrow \quad u = 6 \]

\[ = \frac{1}{2} \int_{6}^{51} \sqrt{u} \, du \]

\[ = \left. \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \right|_{6}^{51} \]

\[ = \frac{1}{3} (51^{3/2} - 6^{3/2}) \]

\[ = \frac{1}{3} (\sqrt[3]{51} - 2\sqrt[3]{6}) \approx 116.505 \ldots \]
(b) \[ \int (3 \sec^2 (r) - 2 \sqrt{r-1}) \, dr = 3 \tan (r) - 2 \cdot \frac{2}{3} (r-1)^{3/2} + C \]
\[= 3 \tan (r) - \frac{1}{3} (r-1)^{3/2} + C. \]

(c) \[ \int_0^1 \frac{\cos(x)}{2 + \sin(x)} \, dx = \int_0^{2+\sin(1)} \frac{1}{u} \frac{\cos(x)}{2+\sin(x)} \, du \]
\[u = 2 + \sin(x) \]
\[du = \cos(x) \, dx \]
\[x = 1 \Rightarrow u = 2 + \sin(1) \]
\[x = 0 \Rightarrow u = 2 \]
\[= \ln \left| u \right| \bigg|_2 ^{2+\sin(1)} \]
\[= \ln (2 + \sin(1)) - \ln(2) \]
\[= \ln \left( 1 + \frac{1}{2} \sin(1) \right) \approx 0.351 \ldots \]

Note: \( 2 + \sin(1) > 0 \)
\( 2 > 0 \)

(d) \[ \int_0^1 e^{2x} \, dx = -\frac{1}{2} \left( e^{-2} - e^0 \right) \]
\[= -\frac{1}{2} \left( e^{-2} - 1 \right) \approx 0.432 \ldots \]
\[ \int_{0}^{\pi/4} \frac{2}{1 + x^2} \, dx = 2 \int_{0}^{\pi/4} \frac{1}{1 + x^2} \, dx \]

\[ = 2 \arctan(x) \bigg|_{0}^{\pi/4} = 2 \left( \arctan\left(\frac{\pi}{4}\right) - \arctan(0) \right) \]

\[ = 2 \arctan\left(\frac{\pi}{4}\right) \approx 1.331 \ldots \]

Note: \( \arctan(x) \equiv \tan^{-1}(x) \)

\[ \int_{0}^{\pi/4} \sec(x) \tan(x) \, dx = \sec(x) \bigg|_{0}^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec(0) \]

\[ = \sqrt{2} - 1 \]

\[ \approx 0.414 \ldots \]

Note: \( \frac{d}{dx} \sec(x) = \sec(x) \tan(x) \)

\[ \int_{0}^{\pi/4} \frac{3t}{t^4 + 1} \, dt = \int_{0}^{\pi/4} \frac{3t}{(t^2)^2 + 1} \, dt = \frac{3}{2} \int_{0}^{\frac{\pi}{4}} \frac{1}{(t^2)^2 + 1} \, dt \]

\[ u = t^2 \]
\[ du = 2t \, dt \]

\[ = \frac{3}{2} \int \frac{1}{u^2 + 1} \, du \]

\[ = \frac{3}{2} \arctan(u) + C \]

\[ = \frac{3}{2} \arctan(t^2) + C \]
4. Area. Graph each region and find the requested area.
   a. Find the area bounded by the graph of \( f(x) = 1 + x^2 \) and the x-axis over the interval \([-1, 1]\).
   b. Find the area bounded by the graph of \( f(x) = 1 - e^x \) and the x-axis over the interval \([0, 1]\).
   c. Find the area bounded by the graph of \( f(x) = \sin(x) \) and the x-axis over the interval \([\pi/2, \pi]\).
   d. Give the area bounded between the x-axis and the graph of \( f(x) = x^2 + 2x - 3 \) over the interval \([-2, 2]\).

\[
\text{Note: } 1 + x^2 \geq 0 \text{ on } [-1, 1].
\]

\[
\text{Area} = \int_{-1}^{1} (1 + x^2) \, dx
\]

\[
= (x + \frac{1}{3}x^3) \bigg|_{-1}^{1}
\]

\[
= (1 + \frac{1}{3}) - (1 - \frac{1}{3})
\]

\[
= \frac{8}{3}
\]

\[
\text{Note: } 1 - e^x \leq 0 \text{ on } [0, 1].
\]

\[
\text{Area} = -\int_{0}^{1} (1 - e^x) \, dx
\]

\[
= -(x - e^x) \bigg|_{0}^{1}
\]

\[
= -[(1 - e) - (0 - 1)]
\]

\[
= e - 2 \approx 0.718\ldots
\]
Find the area bounded by the graph of \( f(x) = \sin(x) \) and the x-axis over the interval \([\pi/2, \pi]\).

\[
\text{Note: } \sin(x) > 0 \text{ on } \left[\frac{\pi}{2}, \pi\right]
\]

\[
\text{}\therefore \text{ Area } = \int_{\frac{\pi}{2}}^{\pi} \sin(x) \, dx
\]

\[
= -\cos(x) \bigg|_{\frac{\pi}{2}}^{\pi}
\]

\[
= -(-1 - 0) = 1.
\]

Give the area bounded between the x-axis and the graph of \( f(x) = x^2 + 2x - 3 \) over the interval \([-2, 2]\).

\[
\text{Area } = \text{Area}(\text{shaded}) + \text{Area}(\text{hatched})
\]

\[
\text{Note: } f(x) \leq 0 \text{ on } [-2, 1]
\]

\[
f(x) > 0 \text{ on } [1, 2]
\]

\[
= -\int_{-2}^{1} (x^2 + 2x - 3) \, dx + \int_{1}^{2} (x^2 + 2x - 3) \, dx
\]

\[
= -\left( \frac{x^3}{3} + x^2 - 3x \right) \bigg|_{-2}^{1} + \left( \frac{x^3}{3} + x^2 - 3x \right) \bigg|_{1}^{2}
\]

\[
= -\left( \frac{1}{3} + 1 - 3 \right) - \left( -\frac{8}{3} + 4 + 6 \right) + \left( \frac{8}{3} + 4 - 6 \right) - \left( \frac{1}{3} + 1 - 3 \right)
\]

\[
= \frac{34}{3}
\]
5. Anti-derivatives.
   a. Give the general anti-derivative for $g(x) = x^3 + 2x - \sqrt{x}$.
   b. $F(x)$ is the anti-derivative for the function $x \sqrt{x^2 + 3}$ that satisfies $F(-1) = 2$. Give $3(F(0) - \sqrt{3}) + 1$.
   c. $F''(x) = x^2 - \frac{2}{\sqrt{x}} + 1, \quad F'(1) = -3$ and $F(1) = 2$. Give $F(x)$.
   d. Find a formula for $f(x)$, given that $2x^3 - 3x^2 + x - 1 = \int_{-1}^{x} f(t) \, dt$.
   e. Suppose $f(x)$ is an anti-derivative of $r(x)$, and $g(x)$ is an anti-derivative of $s(x)$. We are given the data in the table about the functions $f, g, r$ and $s$. \[ \int_{1}^{3} (3r(x) - 2s(x)) \, dx = \]

<p>| | | | | |</p>
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<tr>
<td>$f(x)$</td>
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<td>$r(x)$</td>
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<td>$g(x)$</td>
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<td>$s(x)$</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Give the general anti-derivative for $g(x) = x^3 + 2x - \sqrt{x}$. 
\[
\int (x^3 + 2x - \sqrt{x}) \, dx = \int x^3 \, dx + 2 \int x \, dx - \frac{2}{3} \int x^{3/2} \, dx + C
\]
\[
= \frac{1}{4} x^4 + x^2 - \frac{2}{3} x^{3/2} + C
\]

$F(x)$ is the anti-derivative for the function $x \sqrt{x^2 + 3}$ that satisfies $F(-1) = 2$. Give $3(F(0) - \sqrt{3}) + 1$.

The general anti-derivative of $x \sqrt{x^2 + 3}$ is
\[
\int x \sqrt{x^2 + 3} \, dx = \frac{1}{2} \int \sqrt{x^2 + 3} \cdot 2 \cdot x \, dx = \frac{1}{2} \cdot \frac{2}{3} (x^2 + 3)^{3/2} + C
\]
\[ F(x) = \frac{1}{3} (x^2 + 3)^{3/2} + C \quad \text{and} \quad F(-1) = 2 \]

So, \[ 2 = \frac{1}{3} \left( (-1)^2 + 3 \right)^{3/2} + C \]

\[ 2 = \frac{8}{3} + C \quad \Rightarrow \quad C = -\frac{2}{3} \]

\[ F(x) = \frac{1}{3} (x^2 + 3)^{3/2} - \frac{2}{3} . \]

\[ 3 \left( F(0) - \sqrt{3} \right) + 1 = 3 \left( \left( \frac{1}{3} \cdot 3^{3/2} - \frac{2}{3} \right) - \sqrt{3} \right) + 1 \]

\[ = 3 \left( \sqrt{3} - \frac{2}{3} - \sqrt{3} \right) + 1 \]

\[ = -1 . \]

\[ F''(x) = x^2 - \frac{2}{\sqrt{x}} + 1 , \quad F'(1) = -3 \quad \text{and} \quad F(1) = 2 . \quad \text{Give} \ F(x) . \]

\[ F'(x) = \int \left( x^2 - \frac{2}{\sqrt{x}} + 1 \right) dx = \frac{1}{3} x^3 - 4\sqrt{x} + x + C \]

\[ \quad \text{and} \quad F'(1) = -3 . \]

\[ -3 = \frac{1}{3} - 4 + 1 + C \quad \Rightarrow \quad C = -\frac{1}{3} \]

\[ \Rightarrow \quad F'(x) = \frac{1}{3} x^3 - 4\sqrt{x} + x - \frac{1}{3} \]

\[ \Rightarrow \quad F(x) = \int \left( \frac{1}{3} x^3 - 4\sqrt{x} + x - \frac{1}{3} \right) dx \]

\[ = \frac{1}{12} x^4 - \frac{8}{3} x^{3/2} + \frac{1}{2} x^2 - \frac{1}{3} x + C \]

\[ \quad \text{and} \quad F(1) = 2 \]

\[ \Rightarrow \quad 2 = \frac{1}{12} - \frac{8}{3} + \frac{1}{2} - \frac{1}{3} + C \]

\[ \Rightarrow \quad C = \frac{53}{12} \]

\[ \Rightarrow \quad F(x) = \frac{1}{12} x^4 - \frac{8}{3} x^{3/2} + \frac{1}{2} x^2 + \frac{17}{3} x + \frac{53}{12} . \]
Find a formula for \( f(x) \), given that \( 2x^3 - 3x^2 + x - 1 = \int_{-1}^{x} f(t)dt \).

Differentiate both sides with respect to \( x \).

\[
\frac{d}{dx} \left( 2x^3 - 3x^2 + x - 1 \right) = \frac{d}{dx} \int_{-1}^{x} f(t)dt
\]

\[
6x^2 - 6x + 1 = f(x)
\]

Suppose \( f(x) \) is an anti-derivative of \( r(x) \), and \( g(x) \) is an anti-derivative of \( s(x) \). We are given the data in the table about the functions \( f, g, r \) and \( s \). \( \int_{1}^{3} (3r(x) - 2s(x)) \, dx = \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>( f(x) )</td>
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<td>4</td>
<td>2</td>
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</tbody>
</table>

\[
\int_{1}^{3} (3r(x) - 2s(x)) \, dx = \left( 3f(3) - 2g(3) \right) - \left( 3f(1) - 2g(1) \right)
\]

\[
= \left( 3 \cdot 3 - 2 \cdot 4 \right) - \left( 3 \cdot 1 - 2 \cdot 2 \right)
\]

\[
= (9 - 8) - (3 - 4)
\]

\[
= 1 - 1 = 0
\]
6. **Use the following information in parts a-c.** The graph of \( f(x) \) is shown below, and 
\[ f(-2) = 5 \, . \] The area of region A is 7/3, the area of region B is 34/3, and the area of region C is 7/3.

![Graph of f(x)](image)

a. Give the area of the region bounded between the graph of \( f(x) \) and the x-axis on the interval \([-2, 4]\).

b. \[ \int_{-1}^{4} f(x) \, dx = \]

c. \[ \int_{-1}^{3/2} \left(3x - \frac{d}{dx} f(2x)\right) \, dx = \]

d. The graph of \( y = g'(x) \) is shown below, and \( g(1) = 1 \). Give the values for \( g(0), g(2) \) and \( g(3) \).
Use the following information in parts a-c. The graph of \( f(x) \) is shown below, and \( f(-2) = 5 \). The area of region A is \( 7/3 \), the area of region B is \( 34/3 \), and the area of region C is \( 7/3 \).

a. Give the area of the region bounded between the graph of \( f(x) \) and the x-axis on the interval \([-2, 4]\).

\[
\text{Area} = \text{Area}(A) + \text{Area}(B) + \text{Area}(C) = \frac{7}{3} + \frac{34}{3} + \frac{7}{3} = \frac{48}{3} = 16.
\]

b. \( \int_{-1}^{4} f(x) dx = \int_{-1}^{3} f(x) dx + \int_{3}^{4} f(x) dx \)

\[\text{Notes:} \]
\[\overset{1}{f(x)} \leq 0 \text{ on } [-1, 3] \]
\[\Rightarrow \text{Area}(B) = -\int_{-1}^{3} f(x) dx \]
\[\Rightarrow \int_{-1}^{3} f(x) dx = -\frac{34}{3} \]

\[\overset{2}{f(x)} \geq 0 \text{ on } [3, 4] \]
\[\Rightarrow \text{Area}(C) = \int_{3}^{4} f(x) dx \]
\[\Rightarrow \int_{3}^{4} f(x) dx = \frac{7}{3} \]

\[= -\frac{34}{3} + \frac{7}{3} = -\frac{27}{3} = -9\]
\[
\int_{-1}^{3/2} \left( 3x - \frac{d}{dx}f(2x) \right) dx = \int_{-1}^{3/2} 3x \, dx - \int_{-1}^{3/2} \frac{d}{dx}f(2x) \, dx \\
= \frac{3}{2} x^2 \bigg|_{-1}^{3/2} - f(2x) \bigg|_{-1}^{3/2} \\
= \frac{3}{2} \left( \frac{9}{4} - 1 \right) - \left( f(3) - f(-2) \right) \\
= \frac{15}{8} - 0 + 5 = \frac{55}{8}.
\]

From the graph,

\[f(3) = 0, \quad f(-2) = 5\]

The graph of \( y = g'(x) \) is shown below, and \( g(1) = 1 \). Give the values for \( g(0), \ g(2) \) and \( g(3) \).

\[g(1) - g(0) = \int_{0}^{1} g'(x) \, dx = \int_{0}^{1} (2x - 1) \, dx = \left( x^2 - x \right) \bigg|_{0}^{1} = 0.\]

\[\Rightarrow \quad g(0) = g(1) = 1.\]

\[g(2) - g(1) = \int_{1}^{2} g'(x) \, dx = \int_{1}^{2} (-2x + 3) \, dx = \left( -x^2 + 3x \right) \bigg|_{1}^{2} = -4 + 6 = 2.\]

\[\Rightarrow \quad g(2) = g(1) = 1.\]

\[g(3) - g(2) = \int_{2}^{3} g'(x) \, dx = \int_{2}^{3} 2 \, dx = 3 \quad \Rightarrow \quad g(3) = g(2) + 3 = 4.\]