1. Derivatives I.
   a. Define $f(x) = 3x - x^2$. Use the definition of derivative to give a formula for $f'(x)$.
   b. Define
   $$g(t) = \frac{1}{2t-1}$$
   Use the definition of derivative to give a formula for $g'(t)$.
   c. Let $f(x) = 3x^2 - 2x^3$. Give the value of $\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$.
   d. Let $f(x) = \sin(2x)$. Give the value of $\lim_{x\to 1} \frac{f(x) - f(1)}{x-1}$.

(a) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{3(x+h) - (x+h)^2 - (3x - x^2)}{h}$

"$\frac{0}{0}$" indi form

$= \lim_{h\to 0} \frac{\cancel{3x} + 3h - (x^2 + 2xh + h^2) - \cancel{3x} + x^2}{h}$

$= \lim_{h\to 0} \frac{3h - \cancel{x^2} - 2xh - h^2 + \cancel{x^2}}{h}$

$= \lim_{h\to 0} \frac{3\cancel{h} - 2x\cancel{h} - h^{\cancel{2}}}{\cancel{h}}$

$= \lim_{h\to 0} (3 - 2x - h) = \underline{\underline{3 - 2x}}$. #

(b) Define

$$g(t) = \frac{1}{2t-1}$$

Use the definition of derivative to give a formula for $g'(t)$.

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2(t+h)-1} - \frac{1}{2t-1}}{h}$$

"$\frac{0}{0}$" indi form.

$$= \lim_{h \to 0} \frac{\frac{2t-1-(2(t+h)-1)}{(2(t+h)-1)(2t-1)}}{h}$$

$$\frac{\frac{a}{b}}{c} = \frac{a}{bc}$$

$$= \lim_{h \to 0} \frac{\cancel{2t}\cancel{-1}-\cancel{2t}-2h+\cancel{1}}{h[(2(t+h)-1)(2t-1)]}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(2(t+h)-1)(2t-1)}$$

$$= \lim_{h \to 0} \frac{-2}{(2(t+h)-1)(2t-1)}$$

$$= \frac{-2}{(2t-1)^2}$$

(c) Let $f(x) = 3x^2 - 2x^3$. Give the value of $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Notice $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Also, $f'(x) = 6x - 6x^2$

$\Rightarrow f'(1) = 6 - 6 = 0$.

$\therefore \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = 0$.

(d) Let $f(x) = \sin(2x)$. Give the value of $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$.

Notice that $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$

In general $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ ←

and $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ ←

Also, $f'(x) = \cos(2x) \cdot \frac{d}{dx} 2x = 2\cos(2x)$.

$\therefore f'(1) = 2\cos(2)$

$\Rightarrow \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 2\cos(2)$.

2. Derivatives II. Give the derivative of each function.

a. $f(x) = \tan(x^2+1) - 3x^5 - 2x + 1$

b. $g(x) = \dfrac{-2x^2 - 8x - 1}{x-2}$ ← quotient.

c. $R(t) = \sin(2t) - \sqrt{3t-1} + \dfrac{1}{t^2}$

d. $F(z) = \cos^3(2z-1)\sin(3z)$

e. $G(x) = 5(\sin(2x-3) + 3x^3 - 2x)^4$

(a) $f'(x) = \sec^2(x^2+1) \cdot \dfrac{d}{dx}(x^2+1) - 15x^4 - 2$

$= 2x\sec^2(x^2+1) - 15x^4 - 2$

(b) Use the quotient rule.

$$\frac{d}{dx}\frac{u(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2}$$

$$g'(x) = \frac{(x-2)(-4x-8) - (-2x^2-8x-1)(1)}{(x-2)^2}$$

$$= \frac{-4x^2 - \cancel{8x} + \cancel{8x} + 16 + 2x^2 + 8x + 1}{(x-2)^2}$$

$$= \frac{-2x^2 + 8x + 17}{(x-2)^2}$$

(c) $R(t)=\sin(2t)-\sqrt{3t-1}+\dfrac{1}{t^2} = \sin(2t) - (3t-1)^{1/2} + t^{-2}$

$$R'(t) = \cos(2t)\cdot \frac{d}{dt}2t - \frac{1}{2}(3t-1)^{-\frac{1}{2}}\cdot\frac{d}{dt}(3t-1) + (-2)t^{-3}$$

$$= 2\cos(2t) - \frac{3}{2\sqrt{3t-1}} - \frac{2}{t^3}$$

(d) $F(z)=\underbrace{\cos^3(2z-1)\sin(3z)}_{\text{product}}$

$\cos^3(2z-1) = (\cos(2z-1))^3$

Product rule:

$$\frac{d}{dx}(u(x)v(x)) = u(x)v'(x) + v(x)u'(x)$$

$$F'(z) = \cos^3(2z-1)\frac{d}{dz}\sin(3z) + \sin(3z)\frac{d}{dz}\cos^3(2z-1)$$

$$= \cos^3(2z-1)\cos(3z)\cdot 3 + \sin(3z)\cdot 3\cos^2(2z-1)\underbrace{\frac{d}{dz}\cos(2z-1)}$$

$$= 3\cos^3(2z-1)\cos(3z) + 3\sin(3z)\cos^2(2z-1)\underbrace{(-\sin(2z-1))\cdot 2}$$

$$= 3\cos^3(2z-1)\cos(3z) - 6\sin(3z)\cos^2(2z-1)\sin(2z-1)$$

(e) $G(x) = 5(\sin(2x-3) + 3x^3 - 2x)^4$

$$G'(x) = 20\left(\sin(2x-3) + 3x^3 - 2x\right)^3 \cdot \frac{d}{dx}\left(\sin(2x-3) + 3x^3 - 2x\right)$$

$$= 20\left(\sin(2x-3) + 3x^3 - 2x\right)^3 \cdot \left(\cos(2x-3)\cdot 2 + 9x^2 - 2\right)$$

$$= 20\left(\sin(2x-3) + 3x^3 - 2x\right)^3 \left(2\cos(2x-3) + 9x^2 - 2\right)$$

3. Derivatives III.

   a. Give the y-intercept of the tangent line to the graph of $f(x)=\sin(x)-\cos(x)$ at the point where $x=\pi$.
   b. Give the **largest** value of x for which the derivative of $f(x)=x^3-3x+2$ is 0.
   c. $\frac{d}{dx}\left(\frac{\sin(x)+2x}{\cos(x)}\right)=$
   d. $\frac{d^2}{dx^2}\left(-2x^3+5x^2-7\cos(\pi x)\right)=$
   e. Let $f(x)=-2x^3+5x^2-7\cos(\pi x)$. Give the normal line to the graph of $f''(x)$ at $x=1/2$.
   f. $\frac{d^3}{dx^3}\left(\cos(x)-\sin(x)\right)=$

(a) Tangent Line:

Point $= (\pi, f(\pi)) = (\pi, 1)$

slope $= f'(\pi) = -1$

$f'(x) = \cos(x) - -\sin(x) = \cos(x) + \sin(x)$

$\Rightarrow f'(\pi) = \underline{\underline{-1}}$

$y - 1 = (-1)(x - \pi)$

we find the y-intercept by setting $x=0$ and solving for y.

$y - 1 = \pi \Rightarrow y = \pi + 1$.

$\therefore$ the y-intercept is $(0, \pi+1)$.

(b) Give the **largest** value of x for which the derivative of $f(x)=x^3-3x+2$ is 0.

$$f'(x) = 3x^2 - 3$$

Solve $f'(x) = 0$

$$3x^2 - 3 = 0$$

$$x = \pm 1.$$

$\therefore$ the largest value for which $f'(x) = 0$ is $x = 1$.

quotient.

(c) $\dfrac{d}{dx}\left(\dfrac{\sin(x)+2x}{\cos(x)}\right) =$

$$\boxed{\text{quotient rule.} \quad \frac{d}{dx}\frac{u(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2}}$$

$$= \frac{\cos(x)(\cos(x)+2) - (\sin(x)+2x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\boxed{\cos^2(x)} + 2\cos(x) \boxed{+ \sin^2(x)} + 2x\sin(x)}{\cos^2(x)}$$

$$= \frac{1 + 2\cos(x) + 2x\sin(x)}{\cos^2(x)}$$

d

$$\frac{d^2}{dx^2}\left(-2x^3+5x^2-7\cos(\pi x)\right)=$$

$\nwarrow$ $2^{\underline{nd}}$ derivative.

1. $$\frac{d}{dx}\left(-2x^3+5x^2-7\cos(\pi x)\right)$$

$$= -6x^2+10x-7(-\sin(\pi x))\cdot\pi$$

$$= -6x^2+10x+7\pi\sin(\pi x)$$

2.

$$\frac{d^2}{dx^2}\left(-2x^3+5x^2-7\cos(\pi x)\right)$$

$$=\frac{d}{dx}\left(-6x^2+10x+7\pi\sin(\pi x)\right)$$

$$= -12x+10+7\pi\cos(\pi x)\cdot\pi$$

$$= -12x+10+7\pi^2\cos(\pi x)$$

e Let $f(x) = -2x^3 + 5x^2 - 7\cos(\pi x)$. Give the normal line to the graph of $f''(x)$ at $x = 1/2$.

$$f'(x) = -6x^2 + 10x - 7(-\sin(\pi x)) \cdot \pi$$

$$= -6x^2 + 10x + 7\pi \sin(\pi x)$$

$$f''(x) = -12x + 10 + 7\pi \cos(\pi x) \cdot \pi$$

$$\Rightarrow f''(x) = -12x + 10 + 7\pi^2 \cos(\pi x).$$

Find the normal line to the graph of this function at $x = 1/2$.

Normal Line:

perpendicular to the tangent line

Point $= \left(\frac{1}{2}, f''\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, 4\right)$

Slope $= \dfrac{-1}{f'''\left(\frac{1}{2}\right)} = \dfrac{1}{12 + 7\pi^3}$

$$f''\left(\tfrac{1}{2}\right) = -6 + 10 + 0 = 4$$

$$f'''(x) = -12 + 7\pi^2(-\sin(\pi x)) \cdot \pi$$

$$\Rightarrow f'''\left(\tfrac{1}{2}\right) = -12 - 7\pi^3$$

$\therefore$ Normal line equation:

$$y - 4 = \frac{1}{12 + 7\pi^3}\left(x - \frac{1}{2}\right).$$

f) $\frac{d^3}{dx^3}\left(\cos(x)-\sin(x)\right)= \frac{d^2}{dx^2}\left(-\sin(x)-\cos(x)\right)$

3rd derivative

$$= \frac{d}{dx}\left(-\cos(x)+\sin(x)\right)$$

$$= \sin(x)+\cos(x).$$

4. Chain rule.

a. $G(x)=f(v(x))$, and each of f and v are differentiable functions. Suppose $f(1)=2,\ f(3)=-1,\ f'(1)=-1,\ f'(3)=5,\ v(2)=3$ and $v'(2)=4$. Give $G'(2)$.

b. $G(x)=f(v(x))-3v(x)$, and each of f and v are differentiable functions. Suppose $f(1)=3,\ f(3)=-2,\ f'(1)=1,\ f'(3)=4,\ v(2)=1$ and $v'(2)=3$. Give $G'(2)$.

c. $f(x)=u(v(x))+3v(x)$, and each of u and v are differentiable functions. Also, $u(3)=-1,\ u(2)=2,\ v(3)=4,\ v(2)=3,\ u'(3)=-2,\ u'(2)=3,\ v'(3)=5$ and $v'(2)=4$. Find $f'(2)$.

d. $G(x)=\dfrac{(u(x)+v(x))^2+2x}{3v(x)-1}$, and each of u and v are differentiable functions. Also, $u(3)=-1,\ u(2)=2,\ v(3)=4,\ v(2)=3,\ u'(3)=-2,\ u'(2)=3,\ v'(3)=5$ and $v'(2)=4$. Find $G'(2)$.

a) $G(x)=f(v(x))$.

$\Rightarrow G'(x)=f'(v(x))v'(x)$

$\Rightarrow G'(2)=f'(v(2))\,v'(2)$

$=f'(3)\cdot 4 = 5\cdot 4 = 20.$

b) $G(x)=f(v(x))-3v(x)$, and each of f and v are differentiable functions. Suppose $f(1)=3,\ f(3)=-2,\ f'(1)=1,\ f'(3)=4,\ v(2)=1$ and $v'(2)=3$. Give $G'(2)$.

$G'(x)=f'(v(x))v'(x)-3v'(x)$

$\Rightarrow G'(2)=f'(v(2))v'(2)-3v'(2)$

$=f'(1)\cdot 3-3\cdot 3$

$=1\cdot 3-9=-6.$

(c) $f(x)=u(v(x))+3v(x)$, and each of u and v are differentiable functions. Also, $u(3)=-1$, $u(2)=2$, $v(3)=4$, $v(2)=3$, $u'(3)=-2$, $u'(2)=3$, $v'(3)=5$ and $v'(2)=4$. Find $f'(2)$.

$$f'(x) = u'(v(x))v'(x) + 3v'(x)$$

$$\Rightarrow f'(2) = u'(v(2))v'(2) + 3v'(2)$$

$$= u'(3)\cdot 4 + 3\cdot 4$$

$$= (-2)\cdot 4 + 3\cdot 4 = 4$$

(d) $G(x)=\boxed{\dfrac{(u(x)+v(x))^2+2x}{3v(x)-1}}$ ← quotient, and each of u and v are differentiable functions. Also, $u(3)=-1$, $u(2)=2$, $v(3)=4$, $v(2)=3$, $u'(3)=-2$, $u'(2)=3$, $v'(3)=5$ and $v'(2)=4$. Find $G'(2)$.

$$G'(x) = \frac{(3v(x)-1)\frac{d}{dx}\left[(u(x)+v(x))^2+2x\right] - \left[(u(x)+v(x))^2+2x\right]\frac{d}{dx}(3v(x)-1)}{(3v(x)-1)^2}$$

$$= \frac{(3v(x)-1)\left[2(u(x)+v(x))\cdot(u'(x)+v'(x))+2\right] - \left[(u(x)+v(x))^2+2x\right]\cdot 3v'(x)}{(3v(x)-1)^2}$$

$$\Rightarrow G'(2) = \frac{(3v(2)-1)\left[2(u(2)+v(2))\cdot(u'(2)+v'(2))+2\right] - \left[(u(2)+v(2))^2+4\right]\cdot 3v'(2)}{(3v(2)-1)^2}$$

$v(2)=3$

$u(2)=2$

$u'(2)=3$

$v'(2)=4$

$$= \frac{8\left[10\cdot 7+2\right]-\left[29\right]\cdot 12}{64} = \frac{57}{16}$$

5. Rates I.

   a. Give the rate of change of the surface area of a sphere with respect to its radius when the radius is 2 units.

   b. An object is moving along the graph of $f(x) = x^2$. When it reaches the point (2,4), the x coordinate of the object is decreasing at the rate of 3 units/sec. Give the rate of change of the distance between the object and the point (0,1) at the instant when the object is at (2,4).

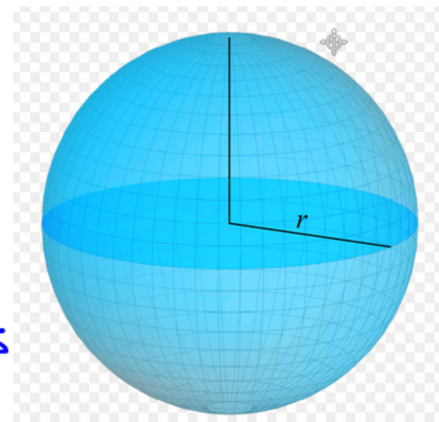
   c. A balloon retains a spherical shape as it is inflated. In addition, the balloon has a volume that is increasing at the constant rate of 1 cm^3/sec. Give the rate of change in the surface area of the balloon when r = 1 cm.

   d. A right circular cone is expanding, and its height and radius are always equal. Also, the volume of the cone is increasing at the rate of 2 cubic inches per minute. How fast is the radius growing when the height is 2 inches?

(a) $S = 4\pi r^2$

$$\Rightarrow \frac{dS}{dr} = 8\pi r$$

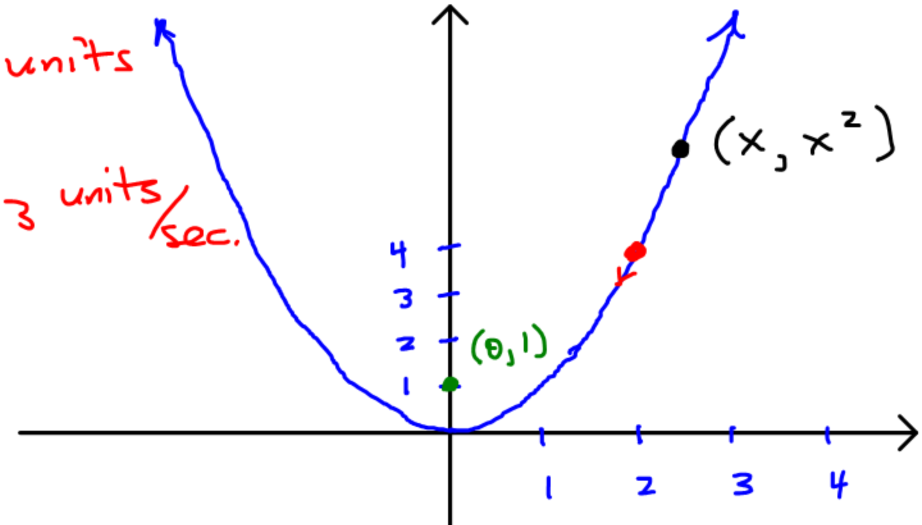
$$\Rightarrow \left.\frac{dS}{dr}\right|_{r=2} = 16\pi \text{ units}$$



(b) An object is moving along the graph of $f(x)=x^2$. When it reaches the point (2,4), the x coordinate of the object is decreasing at the rate of 3 units/sec. Give the rate of change of the distance between the object and the point (0,1) at the instant when the object is at (2,4).

When $x = 2$ units

$\frac{dx}{dt} = -3$ units/sec.



The distance from $(0,1)$ to (x, x^2) is

$$f(x) = \sqrt{x^2 + (x^2-1)^2} = \left(x^2 + (x^2-1)^2\right)^{1/2}$$

Find $\frac{d\, f(x)}{dt}$ when $x = 2$.

$$\frac{d}{dt} f(x) = f'(x)\, \frac{dx}{dt}$$

$$= \frac{1}{2}\left(x^2 + (x^2-1)^2\right)^{-\frac{1}{2}} \cdot \left[2x + 2(x^2-1)\cdot 2x\right] \cdot \frac{dx}{dt}$$

If we substitute $x=2$ and $\frac{dx}{dt} = -3$,

we get

$$\frac{1}{2}(4+9)^{-\frac{1}{2}}\left[4+24\right]\cdot(-3) = \frac{-42}{\sqrt{13}}$$

The rate of change of the distance between the object and the point (0,1) at the instant when the object is at (2,4) is

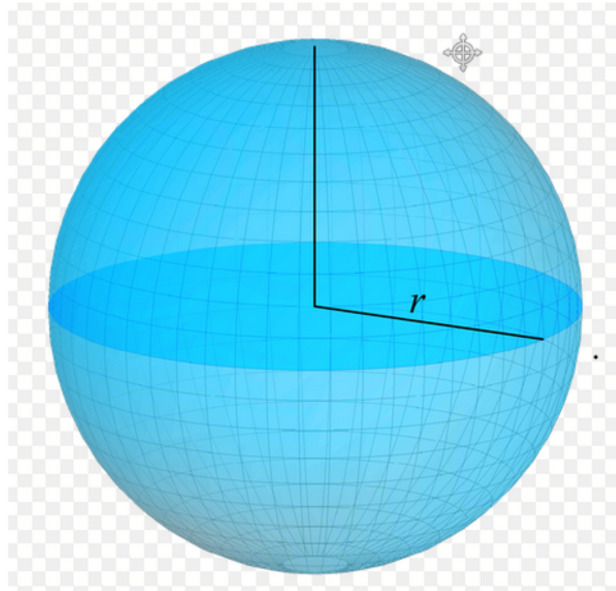
$$\frac{-42}{\sqrt{13}} \text{ units/sec} = \frac{-42\sqrt{13}}{13} \frac{\text{units}}{\text{sec}}.$$

(c) A balloon retains a spherical shape as it is inflated. In addition, the balloon has a volume that is increasing at the constant rate of 1 cm^3/sec. Give the rate of change in the surface area of the balloon when $r = 1$ cm.

$V \equiv$ Volume

$S =$ surface area

$$\frac{dV}{dt} = 1 \ cm^3/sec$$



Find $\frac{dS}{dt}$ when $r = 1$ cm

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$1 = 4\pi \cdot 1^3 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi} \ cm/sec \quad \text{when } r = 1.$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$\therefore$ when $r = 1$, $$\frac{dS}{dt} = 8 \cdot \pi \cdot 1 \cdot \frac{1}{4\pi} \ \frac{cm^2}{sec}$$

$$= 2 \ cm^2/sec.$$

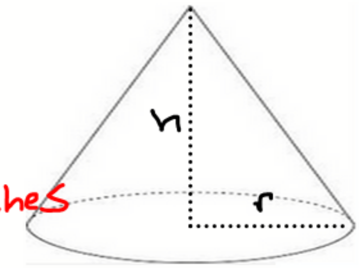
(d) A right circular cone is expanding, and its height and radius are always equal. Also, the volume of the cone is increasing at the rate of 2 cubic inches per minute. How fast is the radius growing when the height is 2 inches?

$V \equiv$ volume

$h \equiv$ height

$r \equiv$ radius

Find $\frac{dr}{dt}$ when $r = 2$ inches



$h = r$ $\qquad \frac{dV}{dt} = 2 \text{ in}^3/\text{min.}$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \qquad \boxed{\text{since } h=r}$$

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}$$

when $r=2$, we know $\frac{dV}{dt} = 2$

$$\Rightarrow \quad 2 = \pi \cdot 2^2 \cdot \frac{dr}{dt}$$

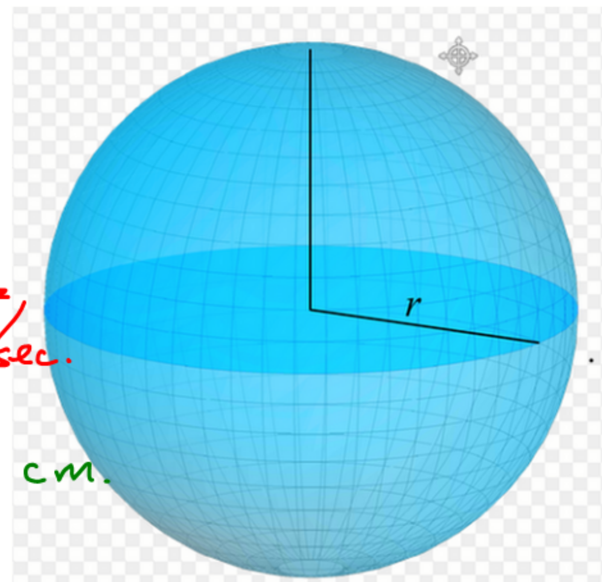
$$\Rightarrow \quad \frac{dr}{dt} = \frac{1}{2\pi} \text{ in/min.}$$

6. Rates II.

   a. The surface area of a sphere is increasing at the rate of 3 cm^2/sec. Give the rate of change of the volume of the sphere when the radius is 2 cm.

   b. A heap of rubbish in the shape of a cube is being compacted. As it compacts, it retains its cubic shape. The change in the width of the cube is given by the function $\frac{dx}{dt} = -\frac{1}{t^2+1}$ in/sec, and $x = 4$ inches when $t = 2$ sec. Give the change in the volume of the cube when $t = 2$ sec.

   c. A right circular cylinder is expanding, and its height and radius are always equal. Also, the volume of the cylinder is increasing at the rate of 1/2 cubic inches per minute. How fast is the surface area growing when the height of the cylinder is 2 inches?



a

V = Volume

S = surface area

r = radius.

We know $\frac{dS}{dt} = 3 \text{ cm}^2/\text{sec.}$

Find $\frac{dV}{dt}$ when $r = 2$ cm.

$V = \frac{4}{3}\pi r^3$ $\qquad$ $S = 4\pi r^2$

$\frac{dV}{dt} = 4\pi \underset{2}{r}^2 \underset{?}{\frac{dr}{dt}}$ $\qquad$ $\underset{3}{\frac{dS}{dt}} = 8\pi \underset{2}{r} \underset{\text{Find}}{\frac{dr}{dt}}$

$3 = 16\pi \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{3}{16\pi} \text{ cm/sec}$

$\Rightarrow$

$\frac{dV}{dt} = 16\pi \cdot \frac{3}{16\pi} \text{ cm}^3/\text{sec}$

$\therefore \frac{dV}{dt} = 3 \text{ cm}^3/\text{sec.}$

6

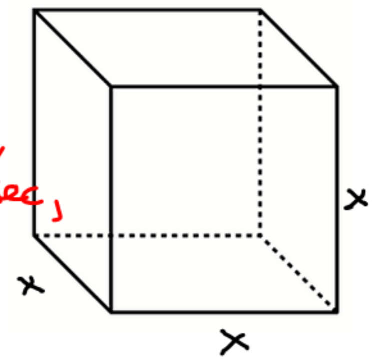
A heap of rubbish in the shape of a cube is being compacted. As it compacts, it retains its cubic shape. The change in the width of the cube is given by the function $\frac{dx}{dt} = -\frac{1}{t^2+1}$ in/sec, and $x = 4$ inches when $t = 2$ sec. Give the change in the volume of the cube when $t = 2$ sec.

V = Volume

x = width

We know $\frac{dx}{dt} = \frac{-1}{t^2+1}$ in/sec,

and $x = 4$ inches when $t = 2$ sec.



Find $\frac{dV}{dt}$ when $t = 2$ sec.

$$V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$\therefore$ since $x = 4$ when $t = 2$, and

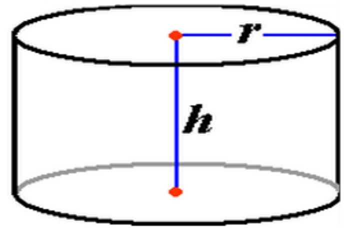
$\frac{dx}{dt} = \frac{-1}{2^2+1}$ when $t = 2$,

$$\frac{dV}{dt} = 48\left(\frac{-1}{5}\right) = \frac{-48}{5} \text{ in}^3/\text{sec}$$

when $t = 2$ sec.

(c)

A right circular cylinder is expanding, and its height and radius are always equal. Also, the volume of the cylinder is increasing at the rate of 1/ 2 cubic inches per minute. How fast is the surface area growing when the height of the cylinder is 2 inches?



V = volume

S = surface area

r = radius

h = height

we are told that $r = h$, and

$$\frac{dV}{dt} = \frac{1}{2} \text{ in}^3/\text{min.}$$

Find $\frac{dS}{dt}$ when $h = r = 2$ inches.

$V = \pi r^2 h = \pi r^3$, $S = 2\pi r^2 + 2\pi r h$

$S = 4\pi r^2$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$\Rightarrow$ when $r = 2$,

$$\frac{1}{2} = 3\pi \cdot 4 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{24\pi} \text{ in}/\text{min}$$

when $r = 2$

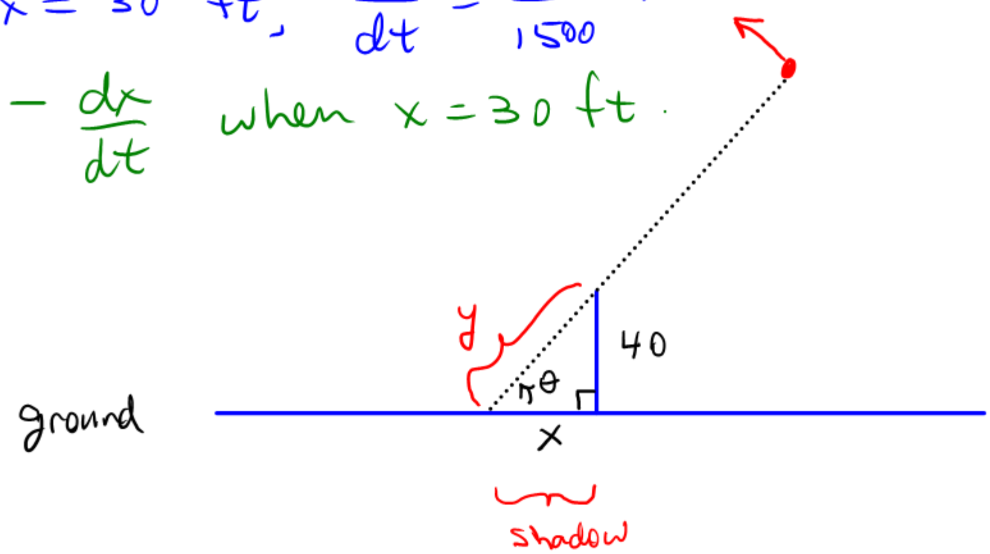
$$\frac{dS}{dt} = 16\pi \cdot \frac{1}{24\pi} \frac{\text{in}^2}{\text{min}}$$

$$= \frac{2}{3} \text{ in}^2/\text{min.}$$

7. Rates III.
    a. On a morning of a day when the sun will pass directly overhead, the shadow of a 40 ft building on level ground is 30 feet long. At the moment in question, the angle the sun makes with the ground is increasing at the rate of $\pi/1500$ radians per minute. At what rate is the shadow length decreasing at this instant?
    b. Aman and Hisson are flying a kite. Aman is releasing the string at the constant rate of 10 ft/sec. At a praticular moment, Hisson is jogging along the ground directly below the kite at the rate of 6 ft/sec, at a distance of 300 ft from Aman. Aman had let out 500 ft prior to that moment. How fast is the kite rising at this moment?
    c. The diameter and height of a right circular cylinder are found at a certain instant to be 10 cm and 20 cm respectively. If the diameter of the cylinder is increasing at the rate of 3 cm/sec, what change in height will keep the volume constant?

(a) When $x = 30$ ft, $\frac{d\theta}{dt} = \frac{\pi}{1500}$.

Find $-\frac{dx}{dt}$ when $x = 30$ ft.



$$\cot(\theta) = \frac{x}{40}$$

Diff wrt t. $\Rightarrow$ $-\csc^2(\theta)\frac{d\theta}{dt} = \frac{1}{40}\frac{dx}{dt}$.

When $x = 30$, $\frac{d\theta}{dt} = \frac{\pi}{1500}$, $y = 50$

$\Rightarrow \csc(\theta) = \frac{50}{40} = \frac{5}{4}$.

Substituting $\Rightarrow$ $-\left(\frac{5}{4}\right)^2 \frac{\pi}{1500} = \frac{1}{40}\frac{dx}{dt}$

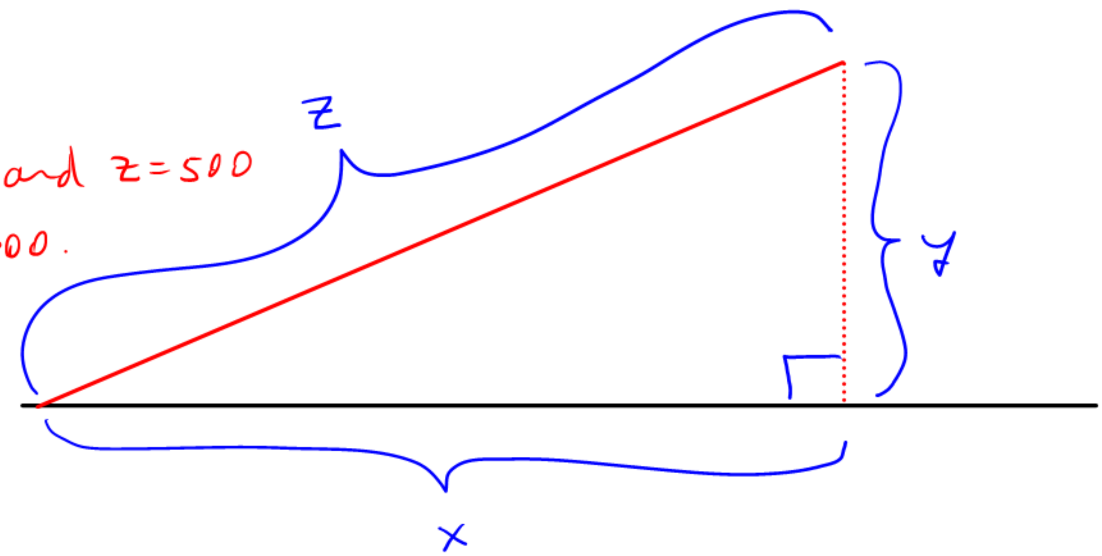
$\Rightarrow \frac{dx}{dt} = \frac{-\pi}{24}$ ft/min

Therefore, at this moment, the shadow length is decreasing at the rate of $\frac{\pi}{24}$ ft/min.

(b)

Aman and Hisson are flying a kite. Aman is releasing the string at the constant rate of 10 ft/sec. At a praticular moment, Hisson is jogging along the ground directly below the kite at the rate of 6 ft/sec, at a distance of 300 ft from Aman. Aman had let out 500 ft prior to that moment. How fast is the kite rising at this moment?

Note:
$x = 300$ and $z = 500$
$\Rightarrow y = 400$.



we know $\dfrac{dz}{dt} = 10$ ft/sec., and

when $x = 300$ft, $\dfrac{dx}{dt} = 6$ ft/sec, and

$z = 500$ ft. Find $\dfrac{dy}{dt}$ when $x = 300$ft.

$$x^2 + y^2 = z^2$$

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt} z^2$$

$$2\underset{300}{x}\,\underset{6}{\frac{dx}{dt}} + 2\underset{400}{y}\,\underset{\text{Find}}{\frac{dy}{dt}} = 2\underset{500}{z}\,\underset{10}{\frac{dz}{dt}}$$

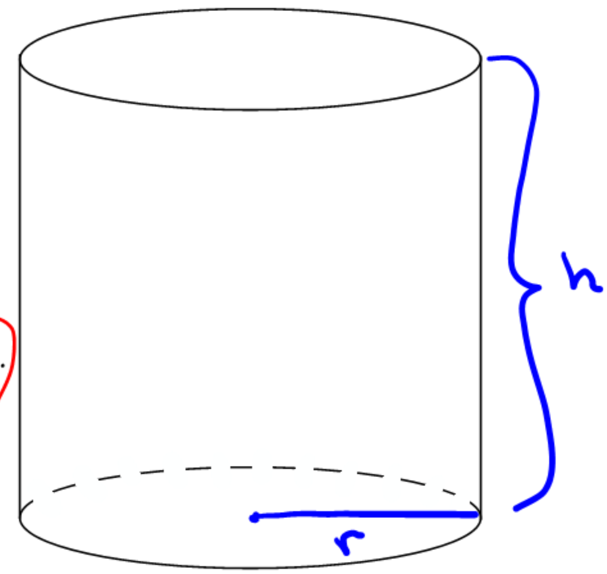
$$\Rightarrow \frac{dy}{dt} = \frac{10{,}000 - 3{,}600}{800} = 8 \text{ ft/sec.}$$

Therefore, the height is increasing at the rate of 8 ft/sec.

(c) The diameter and height of a right circular cylinder are found at a certain instant to be 10 cm and 20 cm respectively. If the diameter of the cylinder is increasing at the rate of 3 cm/sec, what change in height will keep the volume constant?

$r = 5\text{cm}$ and

$h = 20\text{ cm}$ at a particular instance.



Also $\boxed{\frac{d\,2r}{dt} = 3\text{ cm/sec.}}$ $\rightarrow$ $\frac{dr}{dt} = \frac{3}{2}\text{ cm/sec.}$

Find $\frac{dh}{dt}$ so that

$\frac{dV}{dt} = 0.$

$$V = \pi r^2 h$$

Diff. wrt t. $\quad \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + h \cdot 2\pi r \frac{dr}{dt}$

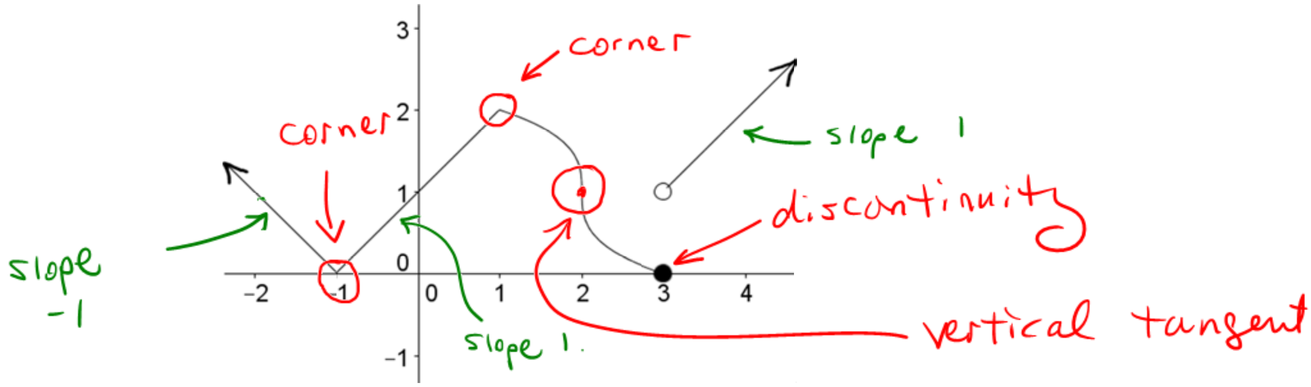
Substitute $\quad 0 = \pi \cdot 25 \frac{dh}{dt} + 40\pi \cdot 5 \cdot \frac{3}{2}$

$$0 = \frac{dh}{dt} + 12 \Rightarrow \frac{dh}{dt} = -12\text{ cm/sec}$$

Therefore, the change in height that will keep the volume constant, at the given instant in time, is

$$-12\text{ cm/sec.}$$

8. The graph of the function $f(x)$ is shown below.



a. Give the values where $f'(x)$ does not exit. Justify your answers.

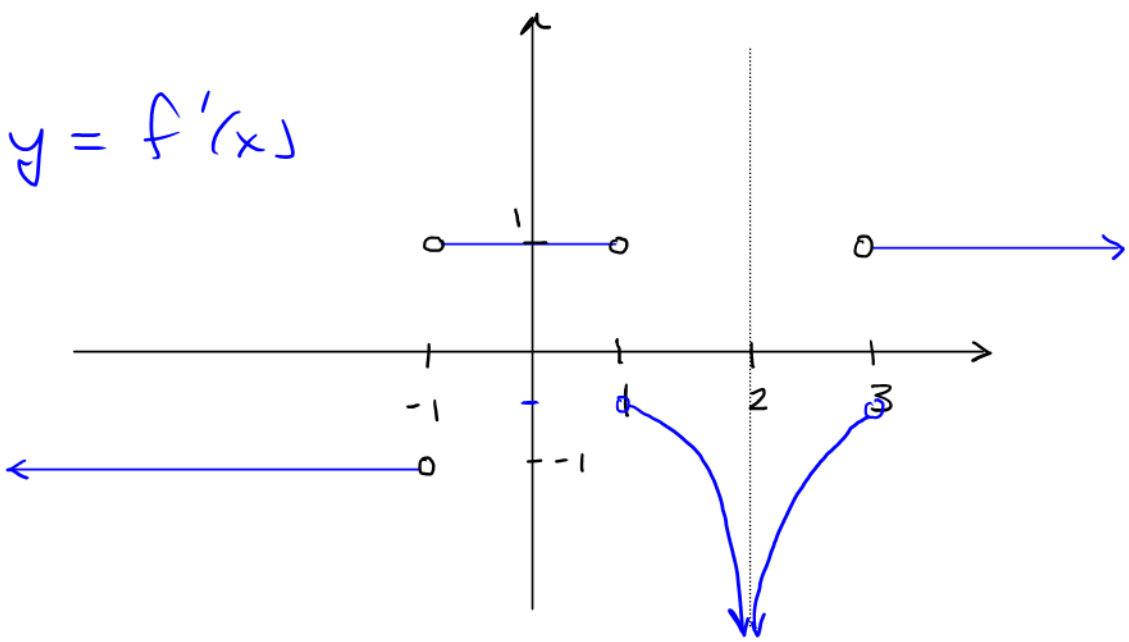
b. Give an approximate sketch of the graph of $f'(x)$.

a) $f'(x)$ does not exist at points where corners, cusps, vertical tangents or discontinuities occur.

$\therefore$ $f'(x)$ does not exist at

$x = -1, 1, 2, 3.$

b) Note: $f'(x)$ is the slope of the graph of $f(x)$ at x.



9. Determine whether each of following statements is **true or false**. Give reasons for your answers.

a. If $f(x)$ is a function and $\lim_{h\to 0}\frac{f(h)-f(-h)}{2h}=0$, then $f'(0)$ exists. False

b. The function $f(x)=\frac{3}{4}(x-1)^{2/3}$ is differentiable at every value of x.

c. The definition of derivative can be used to show that $f(x)=|x|$ is not differentiable at $x=0$.

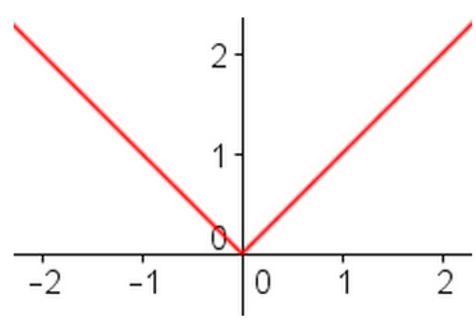
d. $f(x)=x^{1/3}$ is differentiable at $x=0$.

e. $f(x)=\begin{cases} ax^2-x, & x<1 \\ 2, & x=1 \\ bx+\frac{c}{x}, & x>1 \end{cases}$. It is possible to find constants a, b and c so that $f(x)$ is differentiable at $x=1$.

(a) Thought: If f is an even function, then automatically $f(h)-f(-h)=0$.

i.e. $\lim_{h\to 0}\frac{f(h)-f(-h)}{2h}=0$ for every even function.

Note: $f(x)=|x|$ is an even function, and its graph has a corner at $x=0$.



$\therefore$ $f(x)=|x|$ satisfies $\lim_{h\to 0}\frac{f(h)-f(-h)}{2h}=0$, but $f'(0)$ d.n.e.

$\therefore$ The answer is False.

(b) The function $f(x)=\frac{3}{4}(x-1)^{2/3}$ is differentiable at every value of x. False.

Note:

$$f'(x) = \frac{3}{4}\cdot\frac{2}{3}(x-1)^{-1/3}$$

$$= \frac{1}{2(x-1)^{1/3}}.$$

This formula does not make any sense at $x=1$.

Let's check $f'(1) = \lim_{h\to 0} \frac{\overbrace{f(1+h)}^{=\frac{3}{4}h^{2/3}} - \overbrace{f(1)}^{=0}}{h}$

$$= \lim_{h\to 0}\frac{\frac{3}{4}h^{2/3}}{h} = \lim_{h\to 0}\frac{3}{4h^{1/3}} = \text{d.n.e.}$$

Note: $h^{1/3} < 0$ for $h<0$
$h^{1/3} > 0$ for $h>0$
$h^{1/3} \to 0$ as $h \to 0$

$$\lim_{h\to 0^-}\frac{3}{4h^{1/3}} = -\infty$$

$$\lim_{h\to 0^+}\frac{3}{4h^{1/3}} = +\infty$$

$\therefore f'(1)$ does not exist.

So, our statement is False.

(c) The definition of derivative can be used to show that $f(x)=|x|$ is not differentiable at $x=0$. True

Let's check.

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|}{h}$$

$= \text{d.n.e.}$

because

Note: $|h|=\begin{cases}-h, & h<0\\ h, & h>0\end{cases}$

$\therefore \frac{|h|}{h}=\begin{cases}-1, & h<0\\ 1, & h>0\end{cases} \Rightarrow \begin{array}{l}\lim_{h\to 0^-}\frac{|h|}{h}=-1\\ \lim_{h\to 0^+}\frac{|h|}{h}=1.\end{array}$

$\therefore$ The statement is true.

(d) $f(x) = x^{1/3}$ is differentiable at $x = 0$. False.

$f'(x) = \frac{1}{3} x^{-2/3}$, so our formula does not make sense at $x = 0$.

Let's check

$$f'(0) = \lim_{h \to 0} \frac{\overbrace{f(0+h)}^{= h^{1/3}} - \overbrace{f(0)}^{= 0}}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty$$

$\Rightarrow$ $f'(0)$ does not exist.

$\therefore$ the statement is false.

(e) $f(x)=\begin{cases} ax^2-x, & x<1 \\ 2, & x=1 \\ bx+\dfrac{c}{x}, & x>1 \end{cases}$. It is possible to find constants a, b and c so that $f(x)$ is differentiable at $x=1$.

True

Note: If $f'(1)$ exists then f is continuous at $x=1$.

As a result, we expect

$$f(1)=2 = \lim_{x\to 1} f(x)$$

$$\lim_{x\to 1^-}(ax^2-x)=a-1$$

$$\lim_{x\to 1^+}\left(bx+\frac{c}{x}\right)=b+c$$

$\therefore$ If $f'(1)$ exists then

$a-1=2 \Rightarrow \boxed{a=3.}$

and $b+c=2$

Now let's use the def. of deriv.

$a=3$

$f(x)=\begin{cases} ax^2-x, & x<1 \\ 2, & x=1. \\ bx+\dfrac{c}{x}, & x>1 \end{cases}$

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$

Note: $\displaystyle\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$

$$=\lim_{x\to 1^-}\frac{3x^2-x-2}{x-1}$$

$$= \lim_{x\to 1^-} \frac{(3x+2)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x\to 1^-} (3x+2) = \underline{\underline{5}}$$

$\therefore$ $f'(1)$ exists if and only if

$f'(1) = \underline{\underline{5}}$.

Let's check $\displaystyle \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

$\boxed{b+c=2}$

$$= \lim_{x\to 1^+} \frac{bx + \frac{c}{x} - 2}{x-1}$$

$$= \lim_{x\to 1^+} \frac{bx^2 - 2x + c}{x(x-1)}$$

$$= \lim_{x\to 1^+} \frac{(bx-c)\cancel{(x-1)}}{x\cancel{(x-1)}} = b-c$$

$\therefore$ $b - c = 5$ (b/c $f'(1)$ must be 5)

$+$ $(b+c = 2)$

$2b = 7 \Rightarrow b = 7/2$

$\Rightarrow c = -3/2$

$\therefore$ If $a=3$, $b=7/2$ and $c=-3/2$ then $f'(1)$ exists.

$\Rightarrow$ the statement is true.

10. Derivatives and Inverses.

a. The function $f(x) = x^5 + 2x^3 + 3x + 1$ is invertible. Give the y-intercept for the tangent line to the graph of $f^{-1}(x)$ at $x = 1$.

b. Suppose $f(-1) = 3$ and $f'(-1) = 2$. Give the y-intercept of the tangent line to the graph of $f^{-1}(x)$ at $x = 3$.

c. Suppose $(f^{-1})'(1) = 1/3$ and $f(3) = 1$. Give the y-intercept of the normal line to the graph of $f(x)$ at $x = 3$.

If $f(x)$ is invertible and differentiable, then $f^{-1}(x)$ is differentiable. Furthermore, if $f(a) = b$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} \leftarrow \text{provided } f'(a) \neq 0.$$

(a) Tangent Line to $f^{-1}(x)$

point: $(1, f^{-1}(1)) = (1, 0)$

slope: $(f^{-1})'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{3}$

$f^{-1}(1)$? Solve $f(x) = 1$.

$x^5 + 2x^3 + 3x + 1 = 1$

Note: $x = 0$ works.

i.e. $f(0) = 1$

$\therefore f^{-1}(1) = 0$

$f'(x) = 5x^4 + 6x^2 + 3 \Rightarrow f'(0) = 3$

$\therefore$ the tangent line to the graph of $f^{-1}(x)$ at $x = 1$ is $y = \frac{1}{3}(x-1) = \frac{1}{3}x - \frac{1}{3}$

$\therefore$ the y-intercept is $(0, \frac{1}{3})$.

b

Suppose $f(-1)=3$ and $f'(-1)=2$. Give the y-intercept of the tangent line to the graph of $f^{-1}(x)$ at $x=3$.

If $f(x)$ is invertible and differentiable, then $f^{-1}(x)$ is differentiable. Furthermore, if $f(a)=b$, then

$$(f^{-1})'(b)=\frac{1}{f'(a)}$$

we have to assume $f^{-1}(x)$ exists, $f^{-1}(x)$ is differentiable, and $f(x)$ is differentiable.

Tangent Line to $f^{-1}(x)$ at $x=3$:

point $=(3, f^{-1}(3))=(3,-1)$

slope $=(f^{-1})'(3)=\dfrac{1}{f'(-1)}=\dfrac{1}{2}$

$\therefore$ the tangent line to $f^{-1}(x)$ at $x=3$ is given by $y+1=\frac{1}{2}(x-3)$.

$\Rightarrow$ $y=\frac{1}{2}x-\frac{3}{2}-1=\frac{1}{2}x-\frac{5}{2}$.

$\Rightarrow$ the y-intercept is $(0,-5/2)$.

(c) Suppose $\left(f^{-1}\right)'(1)=1/3$ and $f\left(3\right)=1$. Give the y-intercept of the normal line to the graph of $f\left(x\right)$ at $x=3$.

If $f(x)$ is invertible and differentiable, then $f^{-1}(x)$ is differentiable. Furthermore, if $f(a)=b$, then

$$(f^{-1})'(b)=\frac{1}{f'(a)}$$

Normal line:

point $= (3, f(3)) = (3, 1)$

$$\text{slope} = \frac{-1}{f'(3)} = -\frac{1}{f'(3)} = -(f^{-1})'(1)$$

$$= -\frac{1}{3}$$

$\therefore$ the normal line is

$$y-1=-\tfrac{1}{3}(x-3).$$

$$y=-\tfrac{1}{3}x+1+1=-\tfrac{1}{3}x+2.$$

So, the y-intercept is $(0, 2)$.

11. (FR Practice) Let $s(t)=t^3-6t^2$ be the position of a particle in meters along an axis at time t minutes.

a. Give expressions for the velocity and acceleration of the particle.
b. Create a table showing the position, velocity and acceleration of the particle at times $t = 0, 1, 2, 3, 4, 5$ minutes. At each time, indicate whether the particle is speeding up or slowing down.
c. Find all of the times when the particle is at rest.
d. Give all of the values of t when the particle is moving forward.
e. Give all of the values of t when the particle is moving backwards.
f. Give all of the values of t when the particle is speeding up.

g. Give all of the values of t when the particle is slowing down.

(a) velocity $= v(t) = s'(t) = 3t^2 - 12t$

acceleration $= a(t) = v'(t) = 6t - 12$.

(b)

t	$s(t)$	$v(t)$	$a(t)$	speeding up	slowing down
0	0 m	0 m/sec	−12 m/sec²	×	
1	−5 m	−9 m/sec	−6 m/sec²	×	
2	−16 m	−12 m/sec	0 m/sec²		
3	−27 m	−9 m/sec	6 m/sec²		×
4	−32 m	0 m/sec	12 m/sec²		×
5	−25 m	15 m/sec	18 m/sec²	×	

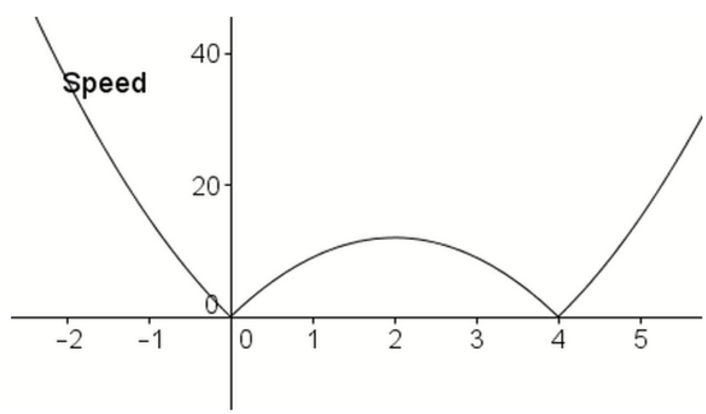
See below.

Note: Speed $= |v(t)| = |3t^2 - 12t|$.

"speeding up" $\iff$ speed is increasing

"slowing down" $\iff$ speed is decreasing

I have graphed "speed" below.



c. Find all of the times when the particle is at rest.

Set $v(t) = 0$ and solve for t.

$$3t^2 - 12t = 0 \iff 3t(t-4) = 0$$

$$\boxed{t = 0 \text{ or } t = 4}$$

(units are in seconds)

d. Give all of the values of t when the particle is moving forward.

"moving forward" $\iff v(t) > 0$.

$$3t^2 - 12t > 0 \iff 3t(t-4) > 0$$

$3t$	$- \; - \; - \; -$	0	$+ \; + \; +$	$+$	$+ \; +$
$t-4$	$- \; - \; -$	$-$	$- \; - \; -$	0	$+ \; + \; +$
t		0		4	

$\therefore$ $3t(t-4) > 0$ iff $t < 0$ or $t > 4$.

i.e. the particle is moving forward iff $t < 0$ or $t > 4$.

e. Give all of the values of t when the particle is moving backwards.

"moving backwards" $\iff v'(t) < 0$.

From the chart in part c, this occurs iff $0 < t < 4$.

i.e. the particle is moving backwards iff $0 < t < 4$.

(f) Give all of the values of t when the particle is speeding up.

The particle is speeding up iff the speed is increasing.

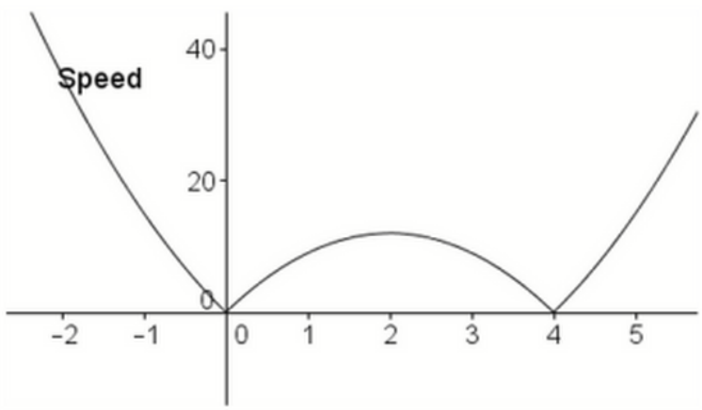
$$\text{Speed} = |v(t)| = |3t^2 - 12t|$$

Note that $3t^2 - 12t < 0$ for $0 < t < 4$

and $3t^2 - 12t > 0$ for $t < 0$ or $t > 4$.

$$\therefore \quad |v(t)| = \begin{cases} 3t^2 - 12t, & t < 0 \\ 12t - 3t^2, & 0 < t < 4 \\ 3t^2 - 12t, & t > 4. \end{cases}$$

The graph was shown in part (b), and is repeated here.



$|v(t)|$ is not differentiable at $t = 0$ or $t = 4$, otherwise,

$$\frac{d}{dt}|v(t)| = \begin{cases} 6t - 12, & t < 0 \\ 12 - 6t, & 0 < t < 4 \\ 6t - 12, & t > 4 \end{cases}$$

$\therefore$ the particle is speeding up when this quantity is positive. i.e. when $0 < t < 2$ or $t > 4$.

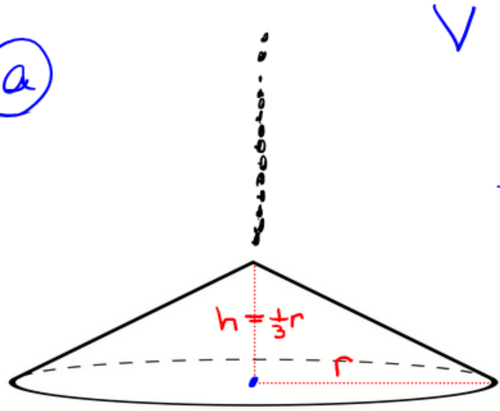
(g) Give all of the values of t when the particle is slowing down.

Similar to (f), the particle is slowing down when $t < 0$ or $2 < t < 4$.

12. (FR Practice) Gravel is being poured by a conveyor onto a pile in such a way that it always retains a conical shape. In addition, the volume of the pile is increasing at the constant rate of 60π cubic feet per minute. Frictional forces cause the height of the pile to always be 1/3 of the radius.

a. Create a detailed diagram illustrating the process described above.
b. How fast is the radius of the pile changing at the instant when the radius is 5 feet?
c. How fast is the height of the pile changing at the instant when the radius is 5 feet?

(a)



V = Volume of the cone in cubic feet.

$\frac{dV}{dt} = 60\pi \text{ ft}^3/\text{min}.$

$h = \frac{1}{3}r$

(b) Find $\frac{dr}{dt}$ when $r = 5$ ft.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\left(\frac{1}{3}r\right)$$

$$\Rightarrow \quad V = \frac{1}{9}\pi r^3$$

So, $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dr}{dt}$

we know $\frac{dV}{dt} = 60\pi \text{ ft}^3/\text{min}$, and

we wand $\frac{dr}{dt}$ when $r = 5$ ft.

Substitution gives

$$60\pi = \frac{1}{3}\pi \cdot 5^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{180}{25} = \frac{36}{5} \text{ ft/min}.$$

when $r = 5$ ft.

(c) How fast is the height of the pile changing at the instant when the radius is 5 feet?

Recall that $h = \frac{1}{3} r$.

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{3} \frac{dr}{dt}.$$

Also, $\frac{dr}{dt} = \frac{36}{5}$ ft/min when $r = 5$ ft.

$$\therefore \quad \frac{dh}{dt} = \frac{1}{3} \cdot \frac{36}{5} = \frac{12}{5} \text{ ft/min}$$

when $r = 5$ ft.

13. (FR Practice) Give complete answers to the following questions.

a. Give the limit definition of the derivative of a function $f(x)$.

b. Use the definition of derivative to give a formula for the derivative of $f(x)=\sqrt{2x+1}$.

c. Use the formula found in part b to give an equation for the tangent line to the graph of $f(x)$ at $x=4$.

d. Use the formula found in part b to find the y-intercept of the normal line to the graph of $f(x)$ at $x=4$.

(a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(b) $$f(x) = \sqrt{2x+1}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$$

$\boxed{\text{"}\frac{0}{0}\text{" ind. form.}}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{2(x+h)+1} - \sqrt{2x+1}\right)\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

Note: $(a-b)(a+b) = a^2 - b^2$

$$= \lim_{h\to 0} \frac{(2(x+h)+1) - (2x+1)}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)} = \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}.$$

(c) Use the formula found in part b to give an equation for the tangent line to the graph of $f(x)$ at $x=4$.

Recall $f(x)=\sqrt{2x+1}$ and $f'(x)=\dfrac{1}{\sqrt{2x+1}}$.

To find the tangent line, we need

point: $(4, f(4)) = (4, 3)$

slope: $f'(4)=\frac{1}{3}$

$\therefore$ an equation for the tangent line is

$$y-3=\frac{1}{3}(x-4).$$

(d) Use the formula found in part b to find the y-intercept of the normal line to the graph of $f(x)$ at $x=4$.

Note: The normal line at $x=4$ is the line perpendicular to the tangent line at $x=4$.

Point: $(4, f(4)) = (4,3)$

slope: $-\dfrac{1}{f'(4)} = -3$

$\therefore$ the normal line is given by

$$y-3=-3(x-4).$$

14. (FR Practice) Give complete answers to the following questions.

a. Verify that $(1,-1)$ is on the graph of $2x^2 + xy + y^3 = x + y$.

b. Give a formula for dy/dx at each point on the graph of $2x^2 + xy + y^3 = x + y$.

c. Find an equation for the tangent line to the graph of $2x^2 + xy + y^3 = x + y$ at the point $(1,-1)$.

d. Give a formula for y'' at each point on the graph of $2x^2 + xy + y^3 = x + y$.

(a) Substituting into the left hand side gives

$$2\cdot(1)^2 + (1)(-1) + (-1)^3$$
$$= 0$$

Substituting into the right hand side gives

$$1 + (-1) = 0.$$

So, both sides are 0.

$\therefore$ $(1,-1)$ is on the graph.

(b) Differentiate both sides of

$$2x^2 + xy + y^3 = x + y$$

wrt x, treating y like a differentiable function of x.

$$\Rightarrow \quad 4x + x\frac{dy}{dx} + y + 3y^2\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \quad (x + 3y^2 - 1)\frac{dy}{dx} = 1 - 4x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-4x-y}{x+3y^2-1}$$

at points (x,y) on the graph.

(c) Find an equation for the tangent line to the graph of $2x^2 + xy + y^3 = x + y$ at the point $(1,-1)$.

we just found $\frac{dy}{dx} = \frac{1-4x-y}{x+3y^2-1}$.

$\therefore \left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{1-4+1}{1+3-1} = \frac{-2}{3}$

So the tangent line at $(1,-1)$ is given by

$$y+1 = -\frac{2}{3}(x-1).$$

(d) Give a formula for y'' at each point on the graph of $2x^2 + xy + y^3 = x + y$.

we know $\frac{dy}{dx} = \frac{1-4x-y}{x+3y^2-1}$.

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1-4x-y}{x+3y^2-1}\right)$$

$$= \frac{(x+3y^2-1)\left(-4+\frac{dy}{dx}\right)-(1-4x-y)\left(1+6y\frac{dy}{dx}\right)}{(x+3y^2-1)^2}$$

$$= \frac{-4(x+3y^2-1) + \frac{dy}{dx}(x+3y^2-1) - (1-4x-y) - 6y\frac{dy}{dx}(1-4x-y)}{(x+3y^2-1)^2}$$

Now, substitute $\frac{dy}{dx} = \frac{1-4x-y}{x+3y^2-1}$ to finish. It won't be pretty. :)