1. Evaluate

composition of continuous functions

a. $\lim_{x\to 3}\sqrt{2x+7} = \sqrt{6+7} = \sqrt{13}$

$2x+7 \geq 0 \quad x \geq -7/2$

Compositions of continuous functions are continuous on open intervals in their domain.

b. $\lim_{x\to -2}(2x^3 - x^2 + 3x - 1) = -16-4-6-1$

polynomial

$= -27.$

Polynomial functions are continuous.

c. $\lim_{x\to 1}\frac{2x-3}{x^2-2x-3} = \frac{-1}{-4} = \frac{1}{4}$

rational function

$\lim_{x\to 1^+}\frac{2x-3}{(x-1)(x+3)} = -\infty$

Rational functions are continuous everywhere they are defined.

"$\frac{-1}{0}$"

d. $\lim_{x\to 1}\frac{2x-3}{x^2+2x-3} = \lim_{x\to 1}\frac{2x-3}{(x-1)(x+3)} = \text{d.n.e.}$

rational function

$\lim_{x\to 1^-}\frac{2x-3}{(x-1)(x+3)} = \infty$

Rational functions are continuous everywhere they are defined.

"$\frac{0}{0}$" indeterminate form

e. $\lim_{x\to 1}\frac{2x-2}{x^2+2x-3} = \lim_{x\to 1}\frac{2\cancel{(x-1)}}{\cancel{(x-1)}(x+3)} = \lim_{x\to 1}\frac{2}{x+3} = \frac{2}{4} = \frac{1}{2}.$

rational function

Rational functions are continuous everywhere they are defined.

f. $\lim_{x\to 1}\frac{x^2-2x-3}{2x-3} = \frac{-4}{-1} = 4$

rational function

Rational functions are continuous everywhere they are defined.

"$\frac{4}{0}$"

g. $\lim_{x\to 1}\frac{|x+3|}{x^2+2x-3} = \text{d.n.e.}$

$|x+3|$ and x^2+2x-3 are continuous.

A quotient of continuous functions is continuous on its domain.

unfortunately $x=1$ is not in the domain.

"$\frac{0}{0}$" indeterminate form

h. $\lim_{x\to -3}\frac{|x+3|}{x^2+2x-3} =$

$|x+3|$ and x^2+2x-3 are continuous.

A quotient of continuous functions is continuous on its domain.

unfortunately $x=-3$ is not in the domain.

"$\frac{4}{0}$" g. $\lim_{x\to 1} \frac{|x+3|}{x^2+2x-3} =$

$|x+3|$ and x^2+2x-3 are continuous.

A quotient of continuous functions is continuous is continuous on its domain.

unfortunately $x=1$ is not in the domain.

Note: $x=1$ is a vertical asymptote for the graph of $\frac{|x+3|}{x^2+2x-3}$.

$$\lim_{x\to 1} \frac{|x+3|}{(x-1)(x+3)} = \lim_{x\to 1} \frac{\cancel{(x+3)}}{(x-1)\cancel{(x+3)}} = \lim_{x\to 1} \frac{1}{x-1}$$

Notes $x+3>0$ for x near 1.

$\therefore$ $|x+3| = x+3$ for x near 1.

Note: $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

$\lim_{x\to 1^+} \frac{1}{x-1} = \infty$

not the same.

$\therefore$ $\lim_{x\to 1} \frac{|x+3|}{x^2+2x-3} =$ d.n.e.

$\lim_{x\to -3} \frac{|x+3|}{x^2+2x-3} = \lim_{x\to -3} \frac{|x+3|}{(x+3)(x-1)}$

$$|u| = \begin{cases} -u, & u < 0 \\ u, & u \geq 0 \end{cases}$$

$$|x+3| = \begin{cases} -(x+3), & x < -3 \\ x+3, & x \geq -3 \end{cases}$$

$$\lim_{x\to -3^-} \frac{|x+3|}{(x+3)(x-1)} = \lim_{x\to -3^-} \frac{-\cancel{(x+3)}}{\cancel{(x+3)}(x-1)}$$

$$= \lim_{x\to -3^-} \frac{-1}{x-1} = \frac{-1}{-4} = \frac{1}{4}$$

$$\lim_{x\to -3^+} \frac{|x+3|}{(x+3)(x-1)} = \lim_{x\to -3^+} \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x-1)}$$

$$= \lim_{x\to -3^+} \frac{1}{x-1} = \frac{1}{-4} = -\frac{1}{4}$$

The left hand limit exists, and the right hand limit exists, but they are not equal.

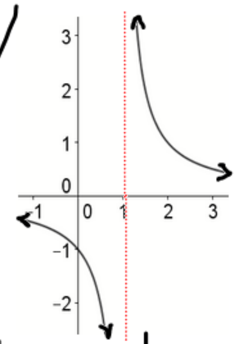
Therefore, the limit does not exist.

$\lim_{x\to -3} \frac{|x+3|}{x^2+2x-3} =$ d.n.e.

2. Evaluate

a. $\lim_{x\to 2}\frac{x+4}{x^2+3x-4} = \frac{6}{6} = 1.$

rational function



Rational functions are continuous on their domain.

"$\frac{5}{0}$" b. $\lim_{x\to 1}\frac{x+4}{x^2+3x-4} = \lim_{x\to 1}\frac{x+4}{(x-1)(x+4)} = \lim_{x\to 1}\frac{1}{x-1}$

rational function $= d.n.e.$

Rational functions are continuous on their domain.
Unfortunately, x=1 is not in the domain of this function. x=1 is a vertical asymptote for the graph.

"$\frac{0}{0}$" indeterminant form c. $\lim_{x\to -4}\frac{x+4}{x^2+3x-4} = \lim_{x\to -4}\frac{\cancel{x+4}}{(\cancel{x+4})(x-1)} = \lim_{x\to -4}\frac{1}{x-1}$

rational function. $= -\frac{1}{5}$

Rational functions are continuous on their domains.
Unfortunately, x=-4 is not in the domain of this function.

d. $\lim_{x\to -\infty}\frac{3x^2+2}{x-2x^2-1} = \lim_{x\to -\infty}\frac{\cancel{x^2}\left(3+\frac{2}{x^2}\right)}{\cancel{x^2}\left(\frac{1}{x}-2-\frac{1}{x^2}\right)}$

$= \lim_{x\to -\infty}\frac{3+\underbrace{2/x^2}_{\to 0}}{\underbrace{1/x}_{\to 0}-2-\underbrace{1/x^2}_{\to 0}} = -\frac{3}{2}$

e. $\lim_{x\to \infty}\frac{3x^2+2}{x-2x^3-1} = \lim_{x\to\infty}\frac{x^2(3+2/x^2)}{x^3(1/x^2-2-1/x^3)}$

$= \lim_{x\to\infty}\underbrace{\frac{1}{x}}_{\to 0}\cdot\frac{(3+\underbrace{2/x^2}_{\to 0})}{\underbrace{1/x^2}_{\to 0}-2-\underbrace{1/x^3}_{\to 0}}$

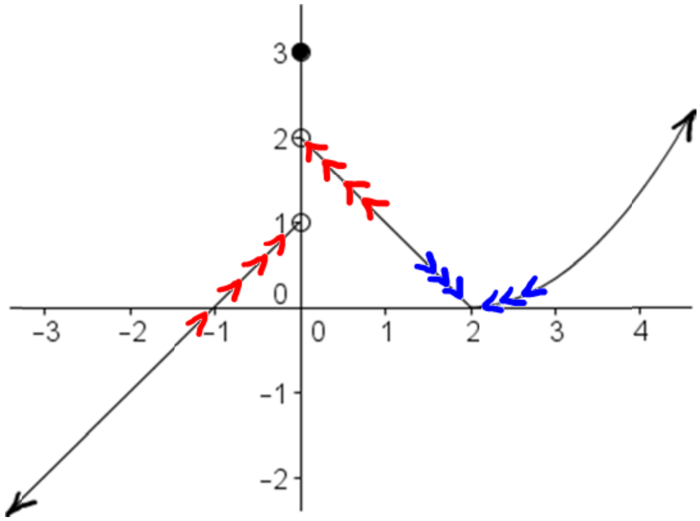
$= 0\cdot\left(-\frac{3}{2}\right) = 0.$

f. $\lim_{x\to -\infty}\frac{3x^3+2}{x-2x^2-1} =$

$= \lim_{x\to -\infty}\frac{x^3(3+2/x^3)}{x^2(1/x-2-1/x^2)}$

$= \lim_{x\to -\infty}\underbrace{x}_{\to -\infty}\cdot\underbrace{\frac{(3+\underbrace{2/x^3}_{\to 0})}{(\underbrace{1/x}_{\to 0}-2-\underbrace{1/x^2}_{\to 0})}}_{\to -\frac{3}{2}} = \infty.$

3. The graph of $y = f(x)$ is shown below.



A jump discontinuity occurs when the limit from the left exists and the limit from the right exists, but they are not the same.

a. $\lim_{x\to 0^-} f(x) =$ 1

b. $\lim_{x\to 0^+} f(x) =$ 2

c. $\lim_{x\to 0} f(x) =$ DNE b/c $1 \neq 2$.

d. $\lim_{x\to 2^-} f(x) =$ 0

e. $\lim_{x\to 2^+} f(x) =$ 0

f. $\lim_{x\to 2} f(x) =$ 0 b/c the LH limit = RH limit = 0

g. Give the interval(s) on which f is continuous, and classify any discontinuities of f.

f is continuous on $(-\infty, 0)$ and $(0, \infty)$.

There is a jump discontinuity at 0.

4. $g(x) = \begin{cases} \boxed{2x - 1}, & x < 2 \\ 3, & x = 2 \\ \boxed{4 - 2x}, & x > 2 \end{cases}$ ← line ← line

a. $\lim_{x \to 2^-} g(x) = \lim_{x\to 2^-} (2x-1) = 3$

← jump at 2.

b. $\lim_{x \to 2^+} g(x) = \lim_{x\to 2^+} (4-2x) = 0$

c. $\lim_{x \to 2} g(x) =$ DNE b/c the LH limit ≠ RH limit

d. Give the interval(s) on which g continuous, and classify any discontinuities of g.

Since $2x-1$ is continuous, g is continuous on $(-\infty, 2)$.

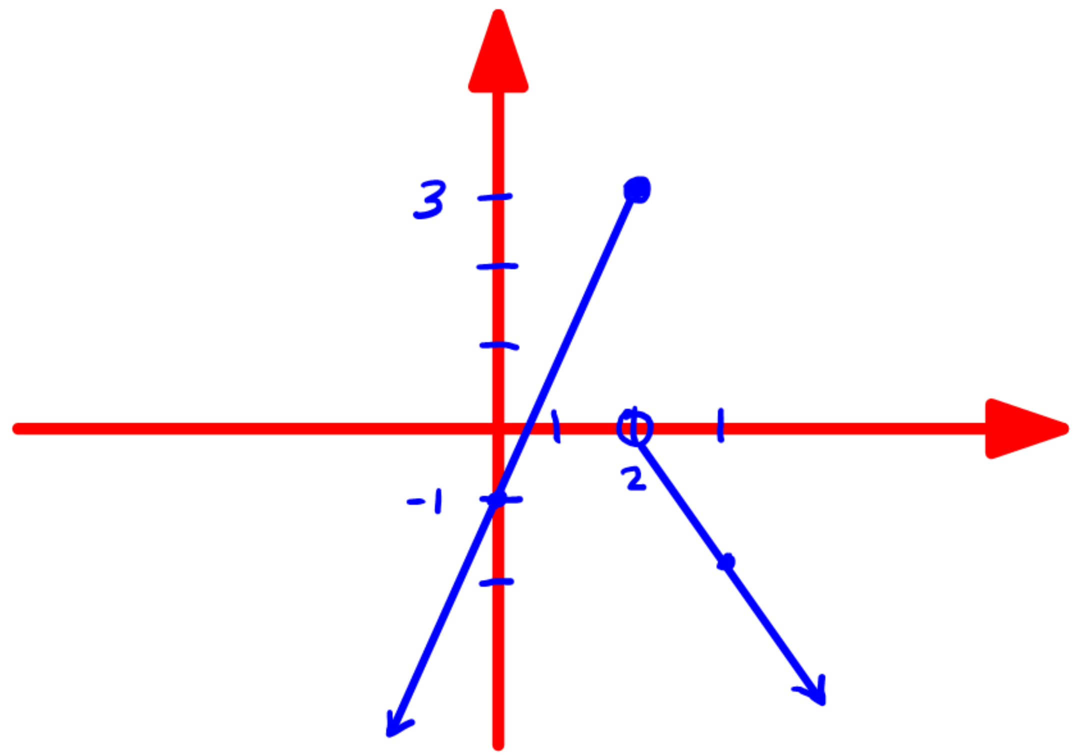
Since $4-2x$ is continuous, g is continuous on $(2, \infty)$.

However $\lim_{x\to 2^-} g(x) = 3$

and $\lim_{x\to 2^+} g(x) = 0$.

Since these exist, but are not equal, g has a jump discont. at $x=2$.

4. $g(x) = \begin{cases} \boxed{2x-1}, & x<2 \\ 3, & x=2 \\ \boxed{4-2x}, & x>2 \end{cases}$ ← line ← line



Update: Since $\lim_{x\to 2^-} g(x) = g(3)$,

$g(x)$ is left continuous at $x=3$.

$\therefore (-\infty, 2]$ and $(2, \infty)$ are intervals of continuity.

5. Evaluate

a. $\lim_{x\to\infty} \frac{x-2^x}{2x^3+2^x} =$ $\lim_{x\to\infty} \frac{\cancel{2^x}\left(\frac{x}{2^x}-1\right)}{\cancel{2^x}\left(\frac{2x^3}{2^x}+1\right)} = -1.$

key fact $\lim_{x\to\infty} \frac{x^p}{2^x} = 0$ if $p>0$.

b. $\lim_{x\to-\infty} \frac{x-2^x}{2x^3+2^x} =$ $\lim_{x\to-\infty} \frac{\cancel{x}\left(1-\frac{2^x}{x}\right)}{\cancel{x^3}\left(2+\frac{2^x}{x^3}\right)}$ $\frac{1}{x^2} \to 0$ $= 0\cdot\frac{1}{2} = 0.$

key fact: $\lim_{x\to-\infty} 2^x = 0$

c. $\lim_{n\to\infty} \frac{12n^4+3n+1}{n^3+2^n} =$ $\lim_{n\to\infty} \frac{\boxed{\frac{n^4}{2^n}}\left(12+\frac{3}{n^3}+\frac{1}{n^4}\right)}{\left(\frac{n^3}{2^n}+1\right)}$ $= 0\cdot 12$ $= 0$

key facts $\lim_{n\to\infty} \frac{n^p}{2^n} = 0$ if $p>0$

d. $\lim_{k\to\infty} \frac{\ln(12k^4+3k+1)}{\sqrt{k}} =$

key fact: $\lim_{k\to\infty} \frac{\ln(k)}{k^p} = 0$ $p>0$

$\rightarrow$ Next page.

e. $\lim_{k\to\infty} \frac{2k^{10}+3^k}{4k^5-3^k} =$ $\lim_{k\to\infty} \frac{\cancel{3^k}\left(\frac{2k^{10}}{3^k}+1\right)}{\cancel{3^k}\left(\frac{4k^5}{3^k}-1\right)} = -1.$

key fact: $\lim_{k\to\infty} \frac{k^p}{3^k} = 0$, $p>0$

$$\lim_{k\to\infty} \frac{\ln(12k^4+3k+1)}{\sqrt{k}} = 0$$

Note: $0 \le \dfrac{\ln(12k^4+3k+1)}{\sqrt{k}}$ for $k > 0$.

$$\le \frac{\ln(16k^4)}{\sqrt{k}} \quad \text{for } k > 1.$$

$$= \frac{\ln(16)}{k^{1/2}} + \frac{4\ln(k)}{k^{1/2}} \quad \text{for } k > 1.$$

$\dfrac{\ln(16)}{k^{1/2}} \to 0$ as $k \to \infty$

$\dfrac{4\ln(k)}{k^{1/2}} \to 0$ as $k \to \infty$

key fact: $\lim_{k\to\infty} \dfrac{\ln(k)}{k^p} = 0, \quad p > 0$

$\lim_{u\to 0} \frac{u}{\sin(u)} = 1$ $\lim_{u\to 0} \frac{\sin(u)}{u} = 1$

6. Evaluate

a. $\lim_{x\to 0} \frac{\sin(x)}{2x} = \frac{1}{2} \boxed{\lim_{x\to 0} \frac{\sin(x)}{x}} = \frac{1}{2}\cdot 1 = \frac{1}{2}$

b. $\lim_{x\to 0} \frac{3x}{\tan(2x)} = \lim_{x\to 0} \frac{3x}{\sin(2x)/\cos(2x)}$

"$\frac{0}{0}$" ind. form

$= \lim_{x\to 0} \underbrace{\boxed{\cos(2x)}}_{\to 1} \underbrace{\boxed{\frac{2x}{\sin(2x)}}}_{\to 1} \cdot 3 \cdot \frac{1}{2} = \frac{3}{2}$

$(a-b)(a+b) = a^2 - b^2$

c. $\lim_{t\to 0} \frac{1-\cos(t)}{t} = \lim_{t\to 0} \frac{(1-\cos(t))(1+\cos(t))}{t\cdot(1+\cos(t))}$

"$\frac{0}{0}$"

you should know this is 0.

$= \lim_{t\to 0} \frac{1-\cos^2(t)}{t(1+\cos(t))} = \lim_{t\to 0} \underbrace{\frac{\sin(t)}{t}}_{\to 1} \underbrace{\frac{\sin(t)}{(1+\cos(t))}}_{\to 0} = 0.$

d. $\lim_{x\to 0} \frac{1-\cos(x)}{2x^2} = \lim_{x\to 0} \frac{1-\cos^2(x)}{2x^2(1+\cos(x))}$

"$\frac{0}{0}$"

$= \lim_{x\to 0} \boxed{\frac{1}{2}} \underbrace{\left(\frac{\sin(x)}{x}\right)^2}_{\to 1} \cdot \underbrace{\frac{1}{1+\cos(x)}}_{\to \frac{1}{2}} = \frac{1}{4}$

e. $\lim_{u\to 0} \frac{3}{u\csc(2u)} = \lim_{u\to 0} \frac{3}{u/\sin(2u)}$

$= \lim_{u\to 0} 2\cdot \underbrace{\boxed{\frac{\sin(2u)}{2u}}}_{\to 1} \cdot 3 = 6$

f. $\lim_{x\to 0} \frac{1-\cos(x)}{\sin^2(2x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{\sin^2(2x)(1+\cos(x))}$

"$\frac{0}{0}$"

Next page.

$= \lim_{x\to 0} \frac{\sin^2(x)}{\sin^2(2x)} \cdot \frac{1}{1+\cos(x)}$

g. $\lim_{x\to 1} \frac{\sin(\pi x)}{x-1} = \lim_{x\to 1} \frac{\sin(\pi(x-1)+\pi)}{x-1}$

"$\frac{0}{0}$"

$= \lim_{x\to 1} \frac{\sin(\pi(x-1))(-1)+0}{x-1}$

$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

$= -\lim_{x\to 1} \pi \cdot \underbrace{\boxed{\frac{\sin(\pi(x-1))}{\pi(x-1)}}}_{\to 1} = -\pi$

$$\lim_{x\to 0}\frac{1-\cos(x)}{\sin^2(2x)} = \lim_{x\to 0}\frac{1-\cos^2(x)}{\sin^2(2x)\left(1+\cos(x)\right)}$$

$$= \lim_{x\to 0}\frac{\sin^2(x)}{\sin^2(2x)}\cdot\frac{1}{1+\cos(x)}$$

$$= \lim_{x\to 0}\left[\underbrace{\frac{\sin(x)}{x}}_{\to 1}\right]^2\cdot\left[\overbrace{\frac{2x}{\sin(2x)}}^{\to 1}\right]^2\cdot\overbrace{\frac{1}{1+\cos(x)}}^{\to \frac{1}{2}}\cdot\frac{1}{4}$$

$$= 1\cdot 1\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$$

7. Evaluate

a. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2}\right) = \lim_{x\to 0} \frac{\boxed{x-2}^{\to -2}}{\boxed{x^2}_{\to 0 \text{ through + values.}}} = -\infty$

$-\infty - \infty = -\infty$

$\infty - \infty = ?$

Note: $x^2 > 0$ for $x \neq 0$

$(a-b)(a+b) = a^2 - b^2$

$3 + x - 3 = x$

b. $\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x} = \lim_{x\to 0} \frac{(\sqrt{3+x}-\sqrt{3})(\sqrt{3+x}+\sqrt{3})}{x(\sqrt{3+x}+\sqrt{3})}$

"$0/0$" ind. form.

$= \lim_{x\to 0} \frac{\cancel{x}\cdot 1}{\cancel{x}(\sqrt{3+x}+\sqrt{3})} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

c. $\lim_{x\to 9^-}\frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9^-} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$

"$0/0$" ind. form

$= \lim_{x\to 9^-} \frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{(x-9)}} = 6$

d. $\lim_{x\to 9^+}\frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9^+} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$

$= \lim_{x\to 9^+} \frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{(x-9)}} = 6$

e. $\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} = 6$ b/c LH Limit = RH Limit.

8. Give the interval(s) of continuity for the function

$$R(x) = \frac{x+4}{x^2+3x-4}$$

and classify any discontinuities of R.

Note: $R(x)$ is a rational function.

R is continuous on its domain.

Domain: All values of x except where $x^2+3x-4=0$.

$$(x+4)(x-1)=0$$

$$x=-4, \quad x=1.$$

$\therefore$ R is continuous on

$$(-\infty,-4) \cup (-4,1) \cup (1,\infty).$$

R is discontinuous at $x=-4$ and $x=1$.

$$R(x) = \frac{x+4}{x^2+3x-4} \quad \text{for all } x \text{ except } x=-4, x=1.$$

$$= \frac{(x+4)}{(x+4)(x-1)}$$

$$= \frac{1}{x-1} \quad \text{except at } x=-4, x=1.$$

Note: $\frac{1}{x-1}$ has a vertical asymptote at $x=1$.

$\therefore$ R has an essential (or infinite) discontinuity at $x=1$.

what about $x = -4$?

$$\lim_{x \to -4} R(x) = \lim_{x \to -4} \frac{1}{x-1} = -\frac{1}{5}$$

So, the limit exists as $x \to -4$

but $R(-4)$ does not exist.

$\therefore$ R has a point (or removable) discontinuity at $x = -4$.

9. Give the interval(s) of continuity for the function

$$F(x) = \frac{|x+3|}{x^2+2x-3}$$

continuous

Quotient

continuous

and classify any discontinuities of F.

The quotient of continuous functions u(x) and v(x) is continuous at every x, except those for which v(x) = 0.

$\therefore$ F is continuous, except where

$$x^2+2x-3=0.$$

$$(x+3)(x-1)=0$$

$$x=-3, \quad x=1$$

i.e. F is continuous on $(-\infty,-3) \cup (-3,1) \cup (1,\infty)$,

The intervals of continuity are $(-\infty,-3)$, $(-3,1)$ and $(1,\infty)$.

and F is discontinuous at $x=-3$ and $x=1$.

$$F(x) = \frac{|x+3|}{x^2+2x-3} = \frac{|x+3|}{(x+3)(x-1)}$$

Note: $|x+3| = \begin{cases} -(x+3), & x<-3 \\ x+3, & x \geq -3 \end{cases}$

$$F(x) = \begin{cases} \dfrac{-(x+3)}{(x+3)(x-1)} & x<-3 \\ \dfrac{(x+3)}{(x+3)(x-1)} & x>-3 \end{cases}$$

$$F(x) = \begin{cases} \dfrac{-(x+3)}{(x+3)(x-1)} & x < -3 \\ \dfrac{(x+3)}{(x+3)(x-1)} & x > -3, \quad x \neq 1 \end{cases}$$

$$= \begin{cases} \dfrac{-1}{x-1}, & x < -3 \\ \dfrac{1}{x-1}, & x > -3, \quad x \neq 1 \end{cases}$$

$$\lim_{x \to -3^-} F(x) = \lim_{x \to -3^-} \frac{-1}{x-1} = \frac{1}{4}$$

$$\lim_{x \to -3^+} F(x) = \lim_{x \to -3^+} \frac{1}{x-1} = -\frac{1}{4}$$

not equal

$\therefore$ since the LH limit exists, the RH limits exists, and they are not equal, F has a jump discont. at $x = -3$.

Near $x = 1$, $F(x) = \dfrac{1}{x-1}$, and $\dfrac{1}{x-1}$ has a vertical asymptote at $x = 1$.

$\therefore$ F has an essential (or infinite) discontinuity at $x = 1$.

10. $g(x) = \begin{cases} \boxed{ax - 1}, & x < 2 \\ 3, & x = 2 \\ \boxed{4 - bx}, & x > 2 \end{cases}$

Give values for a and b so that g is a continuous function.

Regardless of the choices for a and b, $g(x)$ will be continuous for $x < 2$ and $x > 2$.

Focus on $x = 2$.

A function $g(x)$ is continuous at the value $x = 2$ if and only if

1. $g(2)$ exists ✓ $g(2) = 3$
2. $\lim_{x\to 2} g(x)$ exists ✓
3. $\lim_{x\to 2} g(x) = g(2)$ ✓

Note: $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} (ax-1) = 2a-1$

$\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} (4-bx) = 4-2b$

$\therefore$ $\lim\limits_{x\to 2} g(x)$ exists iff $\boxed{2a-1 = 4-2b}$

$\therefore$ $\lim\limits_{x\to 2} g(x) = g(2)$ iff $\boxed{2a-1 = 4-2b}$ and $\boxed{2a-1 = 3}$

$2a-1 = 3 \Leftrightarrow \boxed{a = 2}$

$2a-1 = 4-2b \Leftrightarrow 4-2b = 3 \Leftrightarrow 2b = 1 \Leftrightarrow b = \frac{1}{2}$.

Therefore, g is continuous if and only if $a = 2$ and $b = 1/2$.

11. Determine whether the intermediate value theorem can be used to prove the following equations have solutions on the given interval.

a. $3x^3 - 10x + 1 = 0$, on $[0,1]$.

b. $3x^3 - 5x + \ln(x) = 0$, on $[1,2]$.

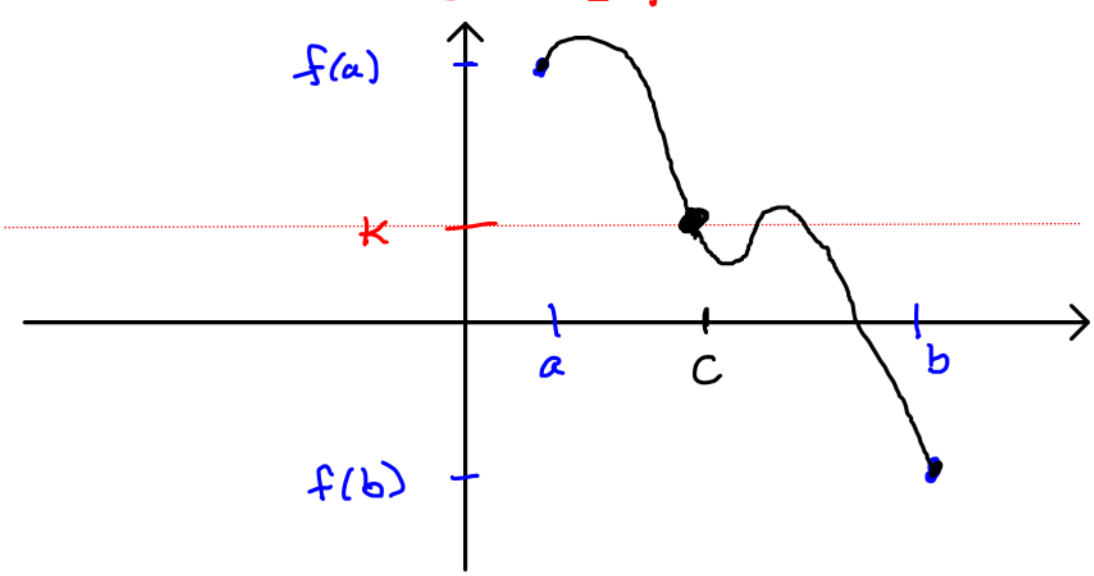
c. $\dfrac{3x^3-10x+1}{x-4} = 0$, on $[2,5]$.

Note: The intermediate value theorem DOES NOT find solutions. It only helps guarantee that there is a solution.

The Intermediate Value Theorem: Suppose $a < b$ and $f(x)$ is a continuous function on the interval $[a, b]$. If K is between $f(a)$ and $f(b)$, then there is a number c between a and b so that $f(c) = K$.

$a < c < b$

This helps us determine if $f(x) = K$ has a solution on $[a,b]$.



a. $\underbrace{3x^3 - 10x + 1} = \overset{k}{0}$, on $[0,1]$.

$f(x)$ is a polynomial.

$\therefore$ $f(x)$ is continuous on $[0,1]$.

Check: $f(0) = 1$

$f(1) = -6$

Note: 0 is between -6 and 1.

$\therefore$ by the intermediate value therem, there is at least one value c so that $0 < c < 1$ and

$$3c^3 - 10c + 1 = 0.$$

Therefore, the intermediate value theorem implies our equation has a solution on the given interval.

b. $3x^3 - 5x + \ln(x) = 0$, on $[1,2]$. k

$$f(x) = \underbrace{3x^3 - 5x}_{\text{continuous everywhere}} + \underbrace{\ln(x)}_{\text{continuous for } x>0}$$

$\therefore$ $f(x)$ is continuous on $[1,2]$.

Check: $f(1) = -2$

$f(2) = 14 + \ln(2) > 14$

(since $\ln(x)$ is increasing and $\ln(1) = 0$)

Note: $-2 < 0 < 14 + \ln(2)$

$\therefore$ by the intermediate value theorem there is a value c so that $1 < c < 2$ and $3c^3 - 5c + \ln(c) = 0$.

Therefore, the intermediate value theorem implies our equation has a solution on the given interval.

c. $\dfrac{3x^3-10x+1}{x-4}=0$, on [2,5].

rational function

$f(x) = \boxed{\dfrac{3x^3-10x+1}{x-4}}$

We cannot use the intermediate value theorem unless f(x) is continuous on the interval [2,5].

Note: $f(x)$ is not defined at $x=4$.

$\therefore$ f is not continuous on $[2,5]$.

$\Rightarrow$ We cannot use the intermediate value theorem to determine whether this equation has a solution on the interval [2,5].

12. $g(x) = \begin{cases} \frac{|x-1|}{x-1}, & x < 1 \\ a, & x = 1 \\ 4 - bx, & x > 1 \end{cases} = \begin{cases} \boxed{-1}, & x < 1 \\ a, & x = 1 \\ \boxed{4 - bx}, & x > 1 \end{cases}$

Give values of a and b so that g is continuous.

$$|x-1| = \begin{cases} -(x-1), & x < 1 \\ x-1, & x > 1 \end{cases}$$

Note that the constant function -1 is continuous, and regardless of the value of b, 4 - bx is a continuous function (since it is a polynomial).

Therefore, we can guarantee that g is continuous by choosing a and b so that g is continuous at x = 1.

1. $g(1)$ exists ✓ $g(1) = a$
2. $\lim_{x \to 1} g(x)$ exists
3. $\lim_{x \to 1} g(x) = g(1)$

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (-1) = -1.$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (4 - bx) = 4 - b.$$

$\therefore \lim_{x \to 1} g(x)$ exists iff $4 - b = -1$. $\Leftrightarrow \boxed{b = 5}$

$-1 = g(1) = a \Leftrightarrow \boxed{a = -1}$

Therefore, the function g is continuous if and only if a = -1 and b = 5.

13. (FR Practice)

$$f(x) = \frac{\boxed{\sin(\pi x)}}{\boxed{x(x+1)}}$$

$\leftarrow$ continuou

$\leftarrow$ continuous

a. Determine the intervals of continuity of f.

b. Classify any discontinuities of f.

c. Determine any horizontal asymptotes of f.

(a) Note that $\sin(\pi x)$ and $x(x+1)$ are continuous for all x.

A quotient of continuous functions is continuous everywhere it is defined.

Set $x(x+1) = 0$

$\Leftrightarrow x = 0, x = -1$

$\therefore f(x)$ is continuous on

$\boxed{(-\infty, -1), (-1, 0) \text{ and } (0, \infty).}$

(b) $f(x)$ is discontinuous at $x=-1$ and $x=0$.

$x=0$: $\lim\limits_{x \to 0} \dfrac{\sin(\pi x)}{x(x+1)} = \lim\limits_{x \to 0} \pi \cdot \underbrace{\boxed{\dfrac{\sin(\pi x)}{\pi x}}}_{\to 1} \cdot \underbrace{\boxed{\dfrac{1}{x+1}}}_{\to 1}$

$= \pi.$

So, $\lim\limits_{x \to 0} f(x)$ exists, but $f(0)$ dne.

$\therefore$ $\boxed{\text{$f(x)$ has a point discontinuity (or removable discontinuity) at $x = 0$.}}$

$\underline{x = -1}$

$$\lim_{x \to -1} \frac{\boxed{\sin(\pi x)}}{x\boxed{(\underline{x+1})}} = \lim_{x \to -1} \frac{\sin(\underline{\pi x + \pi} - \pi)}{x(x+1)}$$

"$\frac{0}{0}$" ind. form

$$= \lim_{x \to -1} \frac{\sin(\overbrace{\pi(x+1)}^{a} - \overbrace{\pi}^{b})}{x(x+1)}$$

$$\lim_{u \to 0} \frac{\sin(u)}{u} = 1$$

sin(a-b) = sin(a)cos(b) - sin(b)cos(a)

$$= \lim_{x \to -1} \frac{\sin(\pi(x+1))\cos(\pi) - 0}{x(x+1)}$$

$$= \lim_{x \to -1} \boxed{\left(-\frac{1}{x}\right)} \cdot \boxed{\frac{\sin(\pi(x+1))}{\pi(x+1)}} \cdot \pi = \pi.$$

$\searrow 1$ $\quad \nearrow 1$

$\underline{\text{Note}}$: $f(-1)$ dne but

$$\lim_{x \to -1} f(x) = \pi$$

$\boxed{\therefore\ f \text{ has a removable (or point) discontinuity at } x = -1.}$

(c) Determine any horizontal asymptotes of f.

$$f(x) = \frac{\sin(\pi x)}{x(x+1)}$$

$-1 \leq \sin(\pi x) \leq 1$

$x(x+1) \to \infty$ as $x \to \pm\infty$

$$\lim_{x \to -\infty} f(x) = 0$$

$$\lim_{x \to \infty} f(x) = 0$$

The line $y = a$ is a horizontal asymptote for f if and only if either

$$\lim_{x \to -\infty} f(x) = a$$

or

$$\lim_{x \to \infty} f(x) = a$$

Therefore, y=0 is the only horizontal asymptote for the function f.

14. (FR Practice) Give complete responses.

a. State the intermediate value theorem.

b. Suppose f is a continuous function on the interval (a, b), and $f(x) \neq 0$ for all $a < x < b$. Show that f is either strictly positive on the entire interval (a, b), or strictly negative on the entire interval (a, b).

c. Use part b to determine the intervals on which

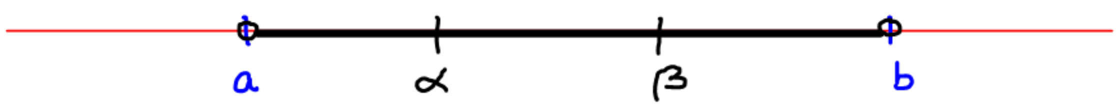
$$f(x) = \frac{x^2 + 3x - 4}{x^2 - 3x - 4}$$

is nonnegative.

(a)

The Intermediate Value Theorem: Suppose $a < b$ and $f(x)$ is a continuous function on the interval $[a, b]$. If K is a number between $f(a)$ and $f(b)$, then there is a number c between a and b so that $f(c) = K$.

(b)



We will show $f(x)$ has the same algebraic sign on (a, b).

Let $\alpha, \beta \in (a, b)$ with $\alpha < \beta$.

If $f(\alpha)$ and $f(\beta)$ have opposite signs then 0 is between $f(\alpha)$ and $f(\beta)$.

Also, f is continuous on $[\alpha, \beta]$.

$\therefore$ there is a value c between α and β so that $f(c) = 0$.

This contradicts the assumption.

$\therefore$ $f(\alpha)$ and $f(\beta)$ have the same algebraic signs. Since α and β are arbitrary, f is either strictly positive or strictly negative on (a,b).

(c) Use part b to determine the intervals on which

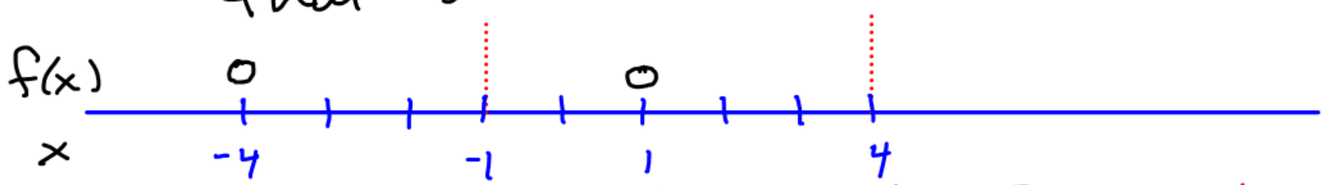
$$f(x) = \frac{x^2 + 3x - 4}{x^2 - 3x - 4} = \frac{(x+4)(x-1)}{(x-4)(x+1)}$$

is nonnegative.

Notes: ① f has vertical asymptotes at $x=4$ and $x=-1$.

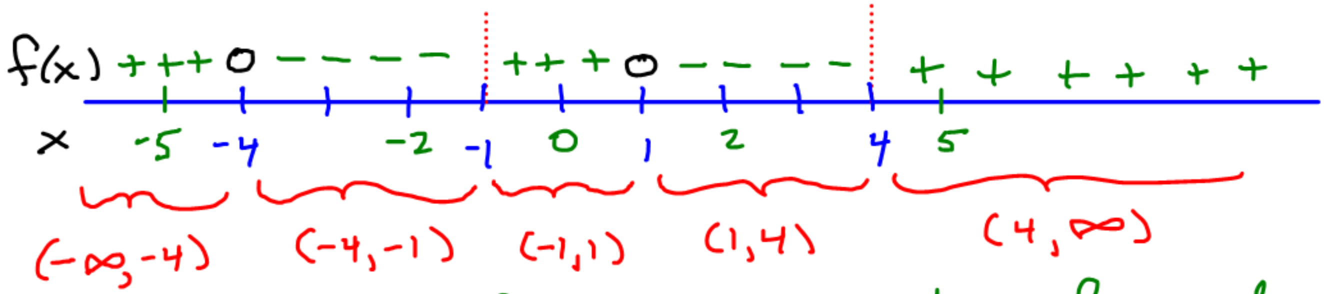
② $f(x)=0$ iff $x=-4$ or $x=1$.

③ f is a rational function, so f is continuous on any interval that does not include 4 or -1.



$(-\infty,-4)$ $(-4,-1)$ $(-1,1)$ $(1,4)$ $(4,\infty)$

f is nonzero and continuous on each of the intervals $(-\infty,-4)$, $(-4,-1)$, $(-1,1)$, $(1,4)$ and $(4,\infty)$. So, from part b, f is either strictly positive or strictly negative on each of these intervals.



choose values from each interval and evaluate f to determine the algebraic sign on the interval.

$$f(x) = \frac{x^2 + 3x - 4}{x^2 - 3x - 4} = \frac{(x+4)(x-1)}{(x-4)(x+1)}$$

$f(-5) > 0 \Rightarrow f(x) > 0$ on $(-\infty, -4)$

$f(-2) < 0 \Rightarrow f(x) < 0$ on $(-4, -1)$

$f(0) > 0 \Rightarrow f(x) > 0$ on $(-1, 1)$

$f(2) < 0 \Rightarrow f(x) < 0$ on $(1, 4)$

$f(5) > 0 \Rightarrow f(x) > 0$ on $(4, \infty)$

$\therefore$ since $f(-4) = 0$ and $f(1) = 0$, f is nonnegative on the intervals $(-\infty, -4]$, $(-1, 1]$ and $(4, \infty)$.

15. (FR Practice) **You are permitted to use a graphing calculator on this problem.**

$$f(x) = \frac{4x - e^x}{x^2 - 2e^x}$$

We must justify our answers! We can use the calculator to find roots and gain insight.

a. Give the domain of f.

b. Find and classify any discontinuities of f.

c. Solve the inequality $f(x) < 0$.

d. Give any horizontal or vertical asymptotes for the graph of f.

(a) Define $u(x) = 4x - e^x$ and $v(x) = x^2 - 2e^x$. $u(x)$ and $v(x)$ are defined for all x. So, $f(x)$ is defined at all x, except where $v(x) = 0$.

we can see one root of $v(x) = 0$ at $x =$ -0.901 ← precise to 3 decimal places.

we will show $v(x)$ is decreasing, and this will verify that there is only one root.

$\boxed{v'(x) = \underline{2x} - \underline{2e^x} < 0 \text{ for } x \le 0}$ ← > 0 for $x \le 0$

≤ 0 for $x \le 0$

$v''(x) = 2 - 2\underline{e^x} < 0$ for $x > 0$

> 1 for $x > 0$

$\Rightarrow v'(x)$ is decreasing for $x \geq 0$.

$\Rightarrow v'(x) < v'(0)$ for $x > 0$.

$\Rightarrow \boxed{v'(x) < -2 \quad \text{for} \quad x > 0.}$

$\Rightarrow v'(x) < 0$ for all x.

$\therefore v$ is decreasing $\Rightarrow$ the root we found is the only root.

$\therefore$ Domain of f :

$(-\infty, -0.901) \cup (-0.901, \infty)$

precise to 3 decimal places.

(b) $f(x) = \dfrac{u(x)}{v(x)}$. From the definitions of $u(x)$ and $v(x)$, we see that both $u(x)$ and $v(x)$ are continuous.

$\therefore f(x)$ is discontinuous where $v(x) = 0$.

From part a, this only occurs at $x = -0.901$

classify : $u(-0.901) = -4.010$

$\Rightarrow u(-0.901) \neq 0$.

$\therefore f(x)$ has a vertical asymptote at $x = -0.901$.

$\Rightarrow$ the discontinuity at $x = -0.901$ is an essential (or infinite) discontinuity.

(c) Solve the inequality $f(x) < 0$.

Recall $f(x) = \dfrac{4x - e^x}{x^2 - 2e^x}$ $\begin{matrix} \leftarrow u(x) \\ \leftarrow v(x) \end{matrix}$

We know $v(x) = 0$ iff $x = -0.901$.

Let's find the root(s) of $u(x)$.

From our graphing calculator, 2 roots are given by

$x = 0.357$ and $x = 2.153$.

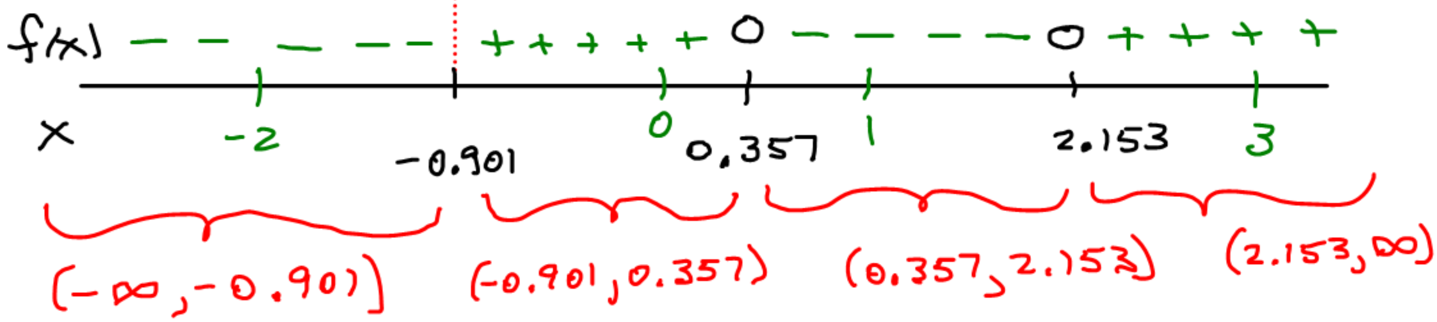
precise to 3 decimal places.

Note: $u'(x) = 4 - e^x$ and $u''(x) = -e^x$

$\Rightarrow$ $u''(x) < 0$ for all x.

$\Rightarrow$ the graph of $u(x)$ is concave down. $\therefore$ the 2 roots above are the only 2 roots of u.

Recall $f(x) = \dfrac{4x - e^x}{x^2 - 2e^x}$ $\begin{matrix} \leftarrow u(x) \\ \leftarrow v(x) \end{matrix}$



f is continuous and nonzero in each of these intervals.

Therefore, we can use the intermediate value theorem to determine the algebraic sign of f(x) in each of these intervals, but simply finding the algebraic sign at select values in each interval.

From our calculator, $f(-2) < 0$, $f(0) > 0$, $f(1) < 0$ and $f(3) > 0$.

$\therefore$ $f(x) < 0$ iff

$-\infty < x < -0.901$ or $0.357 < x < 2.153$.

(d) Give any horizontal or vertical asymptotes for the graph of f.

$$f(x) = \frac{4x - e^x}{x^2 - 2e^x}$$

(d) Give any horizontal or vertical asymptotes for the graph of f.

$$f(x) = \frac{4x - e^x}{x^2 - 2e^x}$$

The only vertical asymptote is at

$x = -0.901$.

for horizontal asymptotes:

$$\lim_{x \to -\infty} \frac{\boxed{4x} - \underbrace{e^x}_{\to 0}}{\boxed{x^2} - 2\underbrace{e^x}_{\to 0}} = 0$$

b/c $\lim_{x \to -\infty} \frac{4}{x} = 0$.

$$\lim_{x \to \infty} \frac{4x - e^x}{x^2 - 2e^x} = \lim_{x \to \infty} \frac{\cancel{e^x}\left(\frac{4x}{e^x} - 1\right)}{\cancel{e^x}\left(\frac{x^2}{e^x} - 2\right)}$$

Note: $\frac{x}{e^x}$ and $\frac{x^2}{e^x} \to 0$

as $x \to \infty$.

$$= \lim_{x \to \infty} \frac{\underbrace{\frac{4x}{e^x}}_{\to 0} - 1}{\underbrace{\frac{x^2}{e^x}}_{\to 0} - 2} = \frac{1}{2}$$

Therefore, there are 2 horizontal asymptotes, given by y = 0 and y = 1/2.