

III. UNION AND INTERSECTION OF EVENTS; COMPLEMENT OF AN EVENT; ODDS

Unions and Intersections:

Suppose we are given an experiment with sample space S . Let A and B be events in S and let E be the event “either A occurs or B occurs”. Then E occurs if the outcome of the experiment is either in A , or in B , or in both A and B . Therefore we conclude that

$$E = A \cup B.$$

Similarly, if we let F be the event “both A and B occur”, then

$$F = A \cap B.$$

Recall that if A and B are sets, then

$$A \cup B = \{x | x \in A \text{ or } x \in B\} \quad \text{and} \quad A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

Example 3.1:

Roll a fair die. The sample space of equally likely simple events is:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Let A be the event “an odd number turns up” and let B be the event “the number that turns up is divisible by 3”.

(a) Find the probability of the event $E =$ “the number that turns up is odd or is divisible by 3”.

(b) Find the probability of the event $F =$ “the number that turns up is odd and is divisible by 3”.

Solutions:

$$(a) A = \{1, 3, 5\}, \quad B = \{3, 6\}, \quad E = A \cup B = \{1, 3, 5, 6\}$$

Since the simple events are equally likely,

$$P(E) = P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{4}{6} = \frac{2}{3},$$

(b) $F = A \cap B = \{3\}$ and

$$P(F) = P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{6}.$$

Suppose that A and B are events, and that $E = A \cup B$. We want to find $P(E)$ in terms of $P(A)$ and $P(B)$? There are two cases to consider:

(i) **A and B are disjoint:** $A \cap B = \emptyset$.

In this case, the events A and B are said to be *mutually exclusive* (if event A occurs, then event B cannot occur and vice versa). For mutually exclusive events A and B , the elements in $A \cup B$ are the elements in A “plus” the elements in B , and

$$(1) \quad P(A \cup B) = P(A) + P(B).$$

(ii) **A and B are not disjoint:** $A \cap B \neq \emptyset$.

In this case, the elements in $A \cup B$ are the elements in A “plus” the elements in B “minus” the elements in $A \cap B$. Therefore,

$$(2) \quad P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Note that the general formula (2) actually holds for both cases: since (2) reduces to (1) when $A \cap B = \emptyset$ [$P(A \cap B) = P(\emptyset) = 0$]. Also note the connection with addition principle of counting:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Examples 3.2:

1. Roll a pair of fair dice.

(a) What is the probability that the sum of the numbers is 7 or 11?

(b) What is the probability that both dice either turn up the same number or that the sum of the numbers is less than 5?

2. What is the probability that a number selected at random from the first 50 positive integers is exactly divisible by 3 or 4?

3. A certain city has 2 daily newspapers, the Chronicle and the Times. A survey of 100 residents of the city was conducted to determine the readership of the two newspapers. It was found that 80 took the Chronicle, 65 took the Times, and 15 took neither paper. Assuming that the survey reflects the actual readership of the city, what is the probability that a resident selected at random:

- (a) Reads at least one of two papers?
 (b) Reads both papers?

Solutions:

1. (a) Let $A =$ “sum is 7” and $B =$ “sum is 11”. Then ,

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} \quad , \quad B = \{(5, 6), (6, 5)\} \quad , \quad \text{and} \quad A \cap B = \emptyset .$$

Therefore,

$$P(A \cup B) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9} .$$

(b) Let $A =$ “both dice turn up the same number” and $B =$ “the sum is less than 5”. Then,

$$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\} \quad , \quad B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\} \quad , \quad \text{and}$$

A and B are not disjoint. Therefore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}$$

2. $S = \{1, 2, 3, \dots, 50\}$ and each number is as likely to be chosen as any other. Let $A =$ “the number is divisible by 3”, $B =$ “the number is divisible by 4”, and $E = A \cup B$.

There are 16 numbers between 1 and 50 that are divisible by 3; $\frac{50}{3} = 16 \frac{2}{3}$;

$$P(A) = \frac{16}{50} .$$

There are 12 numbers between 1 and 50 that are divisible by 4; $\frac{50}{4} = 12 \frac{1}{2}$;

$$P(B) = \frac{12}{50} .$$

Some of these numbers are divisible by both 3 and 4; for example, 12 and 24. The numbers that are divisible by both 3 and 4 are divisible by 12; there are 4 of these;

$$\frac{50}{12} = 4 \frac{1}{6}; \quad P(A \cap B) = \frac{4}{50}.$$

Thus,

$$P(E) = \frac{16}{50} + \frac{12}{50} - \frac{4}{50} = \frac{24}{50} = \frac{12}{25}.$$

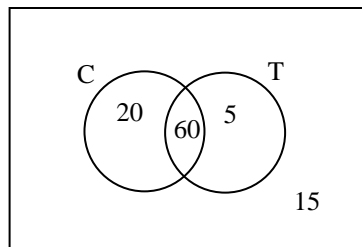
3. (a) Let C = “reads Chronicle” and T = “reads Times”.

Since 15% take neither paper, we conclude that 85% read at least one of the two papers.

Thus,

$$P(C \cup T) = 0.85.$$

(b) From the given information we get the Venn diagram;



and conclude that $P(C \cap T) = 0.60$.

Complement of an Event:

Suppose that E is an event in a sample space S . Since E^c , the *complement of E* , together with E satisfies

$$E \cup E^c = S \quad \text{and} \quad E \cap E^c = \emptyset,$$

it follows that

$$P(S) = P(E \cup E^c) = P(E) + P(E^c).$$

Since $P(S) = 1$ (recall the definition of “probability of an event”), we have

$$P(E) + P(E^c) = 1$$

which implies

$$(3) \quad P(E^c) = 1 - P(E).$$

Examples 3.3:

1. The probability of an event E is $P(E) = 0.63$, what is the probability of the complement of E ?
2. A coin is tossed 4 times. What is the probability of getting at least one tail?
3. In a class of 10 students, 6 are female and 4 are male. If 3 of the students are selected at random, find the probability that at least one female is selected?

Solutions:

$$1. \quad P(E^c) = 1 - P(E) = 1 - 0.63 = 0.37.$$

2. There are 16 equally likely outcomes and $S = \{HHHH, HHHT, HHTH, \dots, TTTT\}$.

Let E = “at least one tail”. Then E^c = “no tails” = “all heads”.

Thus,

$$P(E^c) = \frac{1}{16} \quad \text{and} \quad P(E) = 1 - \frac{1}{16} = \frac{15}{16}.$$

3. Let E = “at least one female is selected”. Then E^c = “no female is selected” = “all males are selected”. There are $C_{10,3} = 120$ combinations of 10 students taken 3 at a time and there are $C_{4,3} = 4$ ways to select 3 male students.

Therefore,

$$P(E^c) = \frac{4}{120} = \frac{1}{30} \quad \text{and} \quad P(E) = 1 - \frac{1}{30} = \frac{29}{30}.$$

Odds

It is common in gaming situations to quote the probability of winning (or losing) in terms of *odds*. The relationship between the odds for (or against) event E and the probability that event E occurs is given by:

$$(i) \quad \text{odds for } E = \frac{P(E)}{1 - P(E)} = \frac{P(E)}{P(E^c)}, \quad \text{provided } P(E) \neq 1;$$

$$(ii) \text{ odds against } E = \frac{P(E^c)}{P(E)}, \quad \text{provided } P(E) \neq 0.$$

When possible, odds are expressed as a ratio of positive integers p/q , (also written $p : q$). In this context, the ratio p/q (or $p : q$) is usually expressed as “ p to q ”.

Examples 3.4:

1. What are the odds for rolling a number divisible by 3 in a single roll of a fair die?
2. What are the odds against rolling a 7 or an 11 in one roll of a pair of fair dice?
3. What are the odds that there is at least one boy in a family of 4 children?

Solutions:

1. Let $E =$ “the number is divisible by 3”. Then $E = \{3, 6\}$; $P(E) = \frac{2}{6} = \frac{1}{3}$

and $P(E^c) = \frac{2}{3}$.

The odds for $E : \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$, 1 to 2, or 1:2.

2. Let $E =$ “7 or 11”. Then, $P(e) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$, and $P(E^c) = \frac{7}{9}$.

The odds against $E : \frac{\frac{7}{9}}{\frac{2}{9}} = \frac{7}{2}$, 7 to 2, or 7:2.

3. Let $E =$ “at least one boy”. Then, $E^c =$ “all girls”, and $P(E^c) = \frac{1}{16}$; $P(E) = \frac{15}{16}$.

The odds for $E : \frac{\frac{15}{16}}{\frac{1}{16}} = \frac{15}{1}$, 15 to 1, or 15:1.

Odds to probability

Suppose we know that the odds for an event E are p to q . Then what is the probability that E occurs? What is $P(E)$?

If the odds for E are p to q , then by the definition of odds given above,

$$\frac{P(E)}{1 - P(E)} = \frac{p}{q}.$$

Therefore,

$$qP(E) = p[1 - P(E)]$$

$$qP(E) = p - pP(E)$$

$$(p + q)P(E) = p$$

and

$$P(E) = \frac{p}{p + q}.$$

Examples 3.5:

1. Calculate the probability of an event E if the odds for E are 5:2.
2. Calculate the probability of an event F if the odds against F are 9:5.

Solutions:

$$1. P(E) = \frac{5}{5+2} = \frac{5}{7}.$$

$$2. \text{The odds for } F \text{ are } 5:9 ; P(F) = \frac{5}{5+9} = \frac{5}{14}$$

Exercises 3.3:

1. A standard die is rolled twice. The outcomes are equally likely. What is the probability that both rolls result in the same number of spots?

2. In the experiment described in Problem 1, what is the probability that the results of the two rolls are different?
3. What is the probability that out of three people selected randomly, at least two will have the same birth month? Assume that all sequences of three birth months are equally likely.
4. In a certain city, 60% of the residents listen either to the opera or the Opry on the radio. Thirty percent of them listen to the opera. Forty four percent of them listen to the Opry. What percentage of the residents listens to both?
5. Suppose that $P(A) = 0.4$, $P(B) = 0.6$ and $P(A \cap B) = 0.1$. Find the probabilities of the events $B^c \cup A^c$ and $A \cap B^c$. A Venn diagram may be helpful.
6. The probability that a certain baseball team will win its division is $2/11$. What are the odds on its winning the division?
7. In a three-horse race, the odds on Blue Boy are 3:7 and the odds on Showgirl are 1:1. What are the odds on Slim Jim?