Similar Polygons

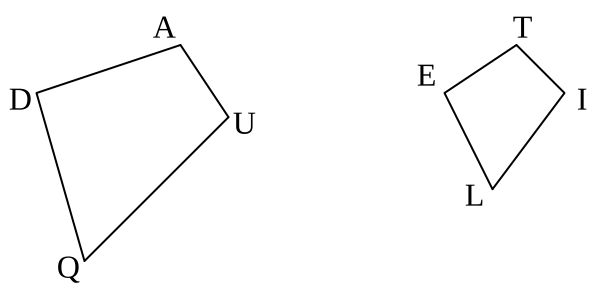
In this unit, we will define similar polygons , investigate ways to show two polygons are similar, and apply similarity postulates and theorems in problems and proofs.

Similar Polygons
Two polygons are similar if and only if their corresponding angles are congruent and there is a proportional relationship among the measures of the corresponding sides.

Quadrilateral QUAD is similar to quadrilateral LITE and can be written as follows using the symbol "~" , which means "is similar to".

QUAD ~ LITE

We will match the corresponding angles and corresponding sides of the two similar polygons below.



The following corresponding angles are congruent:

$\angle Q \cong \angle L, \quad \angle U \cong \angle I, \quad \angle A \cong \angle T, \quad \angle D \cong \angle E$

Corresponding sides of similar polygons have a proportional relationship, which can be written as follows:

$$\frac{DQ}{LE} = \frac{QU}{LI} = \frac{AU}{TI} = \frac{AD}{ET}$$

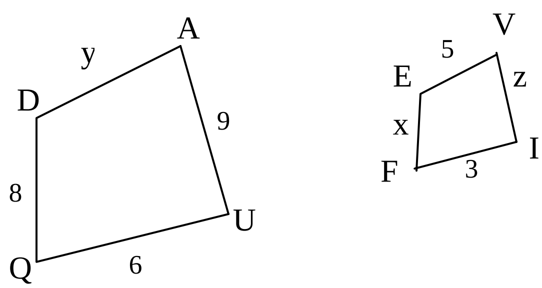
In these ratios, we are comparing the measure of a side of the larger quadrilateral to the corresponding measure of a side of the smaller quadrilateral. We could have compared the measure of a side of the smaller quadrilateral to the corresponding measure of a side of the larger quadrilateral. Order is important in these proportional relationships.

Each of these ratios of corresponding sides determines a <u>scale factor</u> from quadrilateral QUAD to quadrilateral LITE.

Examples

1. Given: Quadrilateral QUAD ~ quadrilateral FIVE

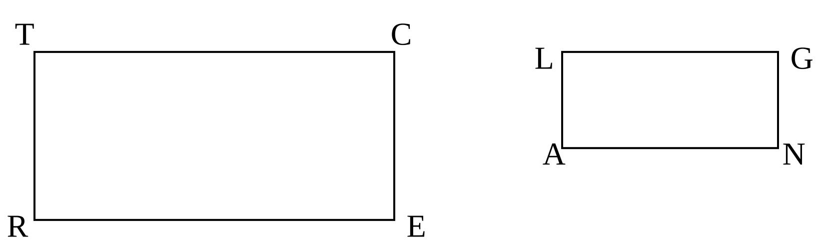
   a) State the congruent angle pairs.
   b) State the scale factor from quadrilateral QUAD to quadrilateral FIVE.
   c) Find *x*, *y*, and *z*.
   d) Find the ratio of the perimeter of quadrilateral QUAD to quadrilateral FIVE.



2. Given: Rectangle RECT ~ rectangle ANGL
   TR:LA = 4:3
   TC = 20 units
   LA = 6 units

   Find the following:

   a) Scale factor from rectangle ANGL to rectangle RECT
   b) LG = _________
   c) Ratio of the perimeter of rectangle RECT to rectangle ANGL



We will now summarize the relationship between the ratio of the measures of any two corresponding sides of similar polygons and the ratio of their corresponding perimeters. This relationship was investigated in the previous examples.

If two polygons are similar, then the ratio of the measures of any two corresponding sides equals the ratio of their corresponding perimeters.

Solutions:

1. a) The following angle pairs are congruent:

$\angle Q \cong \angle F$ $\angle U \cong \angle I$
$\angle A \cong \angle V$ $\angle D \cong \angle E$

b) The *scale factor* from quadrilateral QUAD to quadrilateral FIVE is 2. This is determined by the ratio of QU:FI= 6:3. A *scale factor* of 2 from quadrilateral QUAD to quadrilateral FIVE means that each side of quadrilateral QUAD has a measure that is twice the measure of the corresponding side of quadrilateral FIVE.

If we compare the measures of the sides of quadrilateral FIVE to the measures of the corresponding sides of quadrilateral QUAD, the ratio is 1:2. The *scale factor* from quadrilateral FIVE to quadrilateral QUAD is $\frac{1}{2}$.

c) The *scale factor* can be used to determine each of the missing sides. To find *x* in quadrilateral FIVE, divide the length of its corresponding side of 8 units by 2 or multiply by $\frac{1}{2}$ to get 4 units. The value of *y* in quadrilateral QUAD can be determined by multiplying the measure of its corresponding side of 5 units in quadrilateral FIVE by 2 to get 10 units. The side *z* in the smaller quadrilateral can be found by multiplying the length of its corresponding side in the larger quadrilateral by $\frac{1}{2}$ to get 4.5 units.

d) The perimeter (P_1) of quadrilateral QUAD is the sum of the measures of its sides: $P_1 = (6 + 9 + 10 + 8) = 33$ units

The perimeter (P_2) of quadrilateral FIVE is as follows:
$P_2 = (3 + 4.5 + 5 + 4) = 16.5$ units

The ratio of the perimeters can be expressed as $P_1: P_2 = 33:16.5 = 2:1$. We have determined that the ratio of the perimeters of two similar quadrilaterals is the same as the ratio of the measures of any two corresponding sides.

2. a) The *scale factor* from rectangle ANGL to rectangle RECT is determined by the ratio of the measures of any two corresponding sides of these similar polygons. Since the ratio of corresponding side lengths is given as RT:LA = 4:3, the scale factor from rectangle RECT to rectangle ANGL is $\frac{4}{3}$. The scale factor from rectangle ANGL to rectangle RECT is the reciprocal of $\frac{4}{3}$ or $\frac{3}{4}$. A scale factor of $\frac{3}{4}$ means that the measure of each side of rectangle ANGL is $\frac{3}{4}$ times the measure of each corresponding side in rectangle RECT.

   b) Side $\overline{LG}$ has a measure that is $\frac{3}{4}$ times the measure of its corresponding side length of 20; i.e., LG = $\frac{3}{4} \bullet 20 = 15$ units.

   c) Before finding the perimeter of rectangle RECT, we must first find the actual measure of TR . We are given the ratio TR:LA = 4:3. The ratio 4:3 does not necessarily represent the ratio of the actual measures of the sides. This ratio does represent the ratio of the measures of any two corresponding sides of the rectangles when comparing the side length of the large rectangle to the corresponding side length of the small rectangle. By solving the following proportion, we can find TR.

$$\frac{TR}{LA} = \frac{TC}{LG}$$

$$\frac{TR}{6} = \frac{20}{15}$$

$$\frac{TR}{6} = \frac{4}{3}$$

We can use the scale factor method to solve the proportion by "scaling up" both 4 and 3 in the second ratio by 2 as follows.

$$\frac{TR}{6} = \frac{4}{3} \bullet \frac{2}{2}$$

It follows that TR = 8 units.

The perimeter of rectangle RECT can be found by adding the lengths of the four sides.

$P_{RECT} = (8 + 20 + 8 + 20) = 56$ units

The perimeter of rectangle ANGL can also be found in the same way.

$P_{ANGL} = (6 + 15 + 6 + 15) = 42$ units

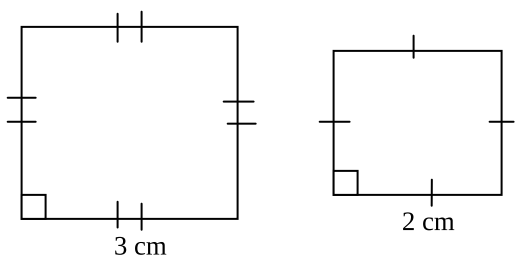
The ratio of the perimeter of rectangle RECT to rectangle ANGL is as follows: 56 units : 42 units or 4:3

The ratio of corresponding perimeters of similar rectangles is the same as the ratio of the measures of any two corresponding sides.

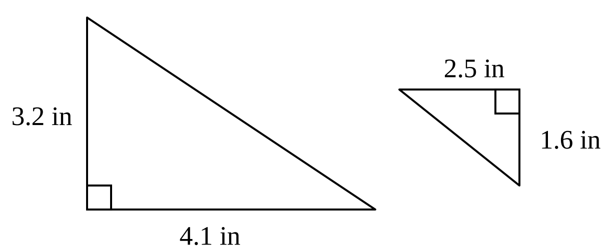
Exercises

1. Determine which of the following pairs of polygons is similar and justify each answer.

a)



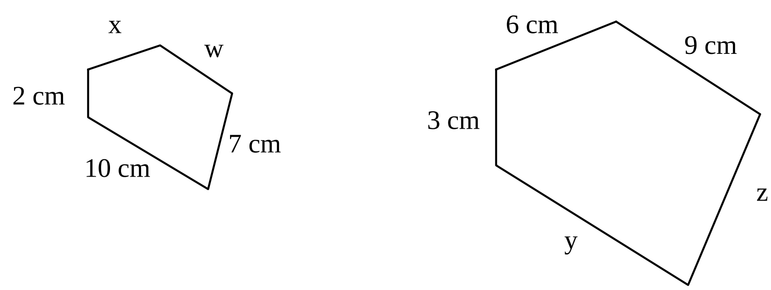
b)



2. Determine which of the following is *True* or *False* and justify each answer.

   a) All squares are similar.
   b) Any two rectangles are similar.
   c) All equilateral triangles are similar.
   d) A regular pentagon with a side length of 5 cm and a regular pentagon with a side length of 3 cm are similar.
   e) Any two parallelograms.
   f) A right triangle with legs of 3 ft and 4 ft and a right triangle with legs of 6 cm and 8 cm are similar.
   g) Any two isosceles triangles are similar.

3. Find the length of each missing side in the two similar polygons below.



4. The ratio of the perimeters of two hexagons is 5:4. What is the ratio of the corresponding sides?

5. Triangle ABC is similar to triangle DEF. The scale factor from triangle ABC to triangle DEF is $\frac{1}{3}$. If DE = 6 m, find the length of $\overline{AB}$.

Similar Triangles

In this unit, we will investigate ways to show two triangles are similar and apply the similarity postulate and theorems in problem situations.

Exploration

Use a TI-83+ graphing calculator with Cabri Junior™ for the following exploration.

1. Is it possible to show two triangles are similar using only two angles of one triangle congruent to two angles of a second triangle?

   Make a conjecture: ____

2. Test this conjecture as follows:
   a) Press F2 and select **Triangle**. Draw and label ΔABC.
   b) Press F2 and select **Segment.** Draw and label $\overline{DE}$.
   c) Copy ∠CAB at point D on $\overline{DE}$ as shown in figure 1.

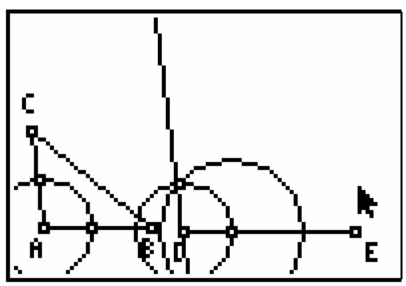


figure 1

   d) Copy ∠CBA at point E on $\overline{DE}$ as shown in figures 2 and 3 below.

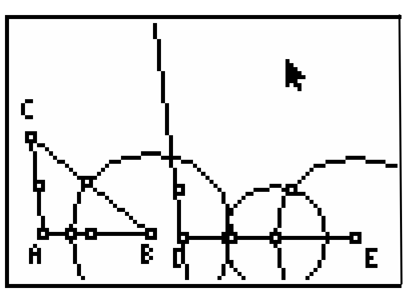


figure 2

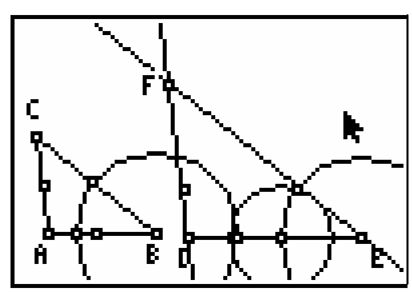


figure 3

e) Press F5 and select **Hide/show-objects**. Hide the construction circles and extra points on the figures as shown in figure 4 below.

f) Press F5 and select **Measure-D. & Length**. Measure the length of each side in ΔABC and ΔDEF as shown in figure 5.

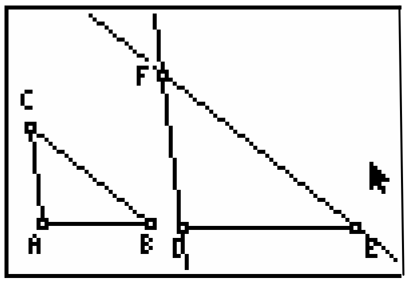


figure 4

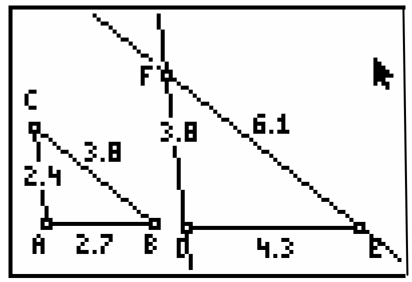


figure 5

g) Find the ratio of corresponding sides and round to the nearest tenth as follows.

$\frac{2.7}{4.3} \approx 0.6$ $\frac{2.4}{3.8} \approx 0.6$ $\frac{3.8}{6.1} \approx 0.6$

Since these ratios of corresponding sides are the same (rounded to the nearest tenth), the corresponding sides of the triangles have a proportional relationship. Both triangles have two angles congruent by construction and the third angles are also congruent (*If two angles of one triangle are congruent to two angles of a second triangle, the third angles are congruent.)* Therefore, ΔABC ~ ΔDEF by the definition of similar polygons.

3. The results of this exploration lead to the following postulate for similar triangles.

Angle-Angle Similarity Postulate

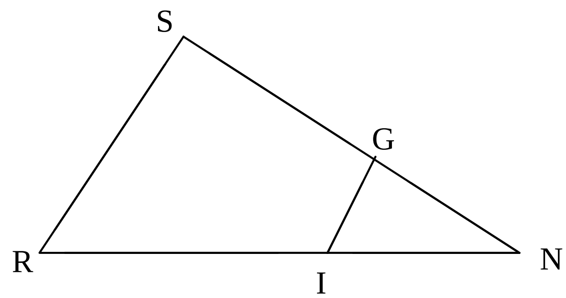
Two triangles are similar if two angles of one triangle are congruent to two angles of a second triangle.

This postulate can be written as the "***AA similarity postulate***" for triangles.

Examples

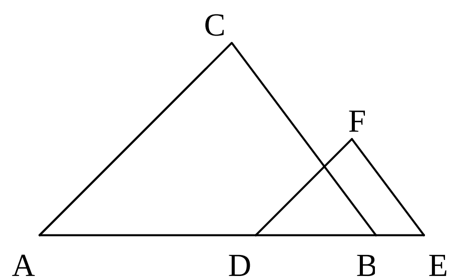
1. Given: $\Delta ING \sim \Delta RNS$
   GS = 6 in
   GN = 3 in
   RI = (x + 5) in
   IN = (x + 1) in

   Find the value of *x*.



2. Given: $\overline{AC} \parallel \overline{DF}$
   $\overline{BC} \parallel \overline{EF}$
   AC = (3x + 2) cm
   DF = (x +2.8) cm
   AD = 6 cm
   DB = 4 cm
   BE = 2 cm

   Find the value of *x* in the figure.



<u>Solutions:</u>

1. Since $\Delta ING \sim \Delta RNS$, corresponding sides have a proportional relationship.
   We can write the following proportion: $\frac{NG}{NS} = \frac{NI}{NR}$

   By substitution, we have the following: $\frac{3}{9} = \frac{(x+1)}{(2x+6)}$

   By simplifying $\frac{3}{9}$, we get the following; $\frac{1}{3} = \frac{(x+1)}{(2x+6)}$

   If $\frac{a}{b} = \frac{c}{d}$, then ad = bc. 3(x + 1) = 1(2x + 6)

   By the distributive property, we have: 3x + 3 = 2x + 6

   Solving for *x*, we get the following: x = 3

2. Since $\overline{AC} \parallel \overline{DF}$, ∠CAB ≅ ∠FDE (*If two parallel lines are cut by a transversal, then the corresponding angles are congruent.*). ∠CBA ≅ ∠FED because $\overline{BC} \parallel \overline{EF}$ and the corresponding angles formed are congruent. Now we have ΔABC ~ ΔDEF by the AA similarity postulate. Since corresponding sides of similar polygons have a proportional relationship, we can write the following proportion and solve for *x*.

$$\frac{AC}{DF} = \frac{AB}{DE}$$

$$\frac{(3x+2)}{(x+2.8)} = \frac{10}{6}$$

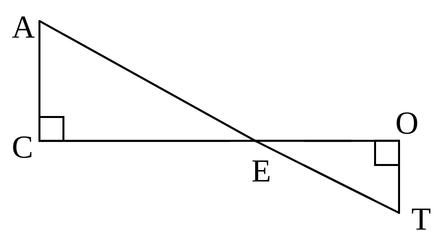
$$\frac{(3x+2)}{(x+2.8)} = \frac{5}{3}$$

$$3(3x + 2) = 5(x + 2.8)$$
$$9x + 6 = 5x + 14$$
$$4x = 8$$
$$x = 2$$

Exercises

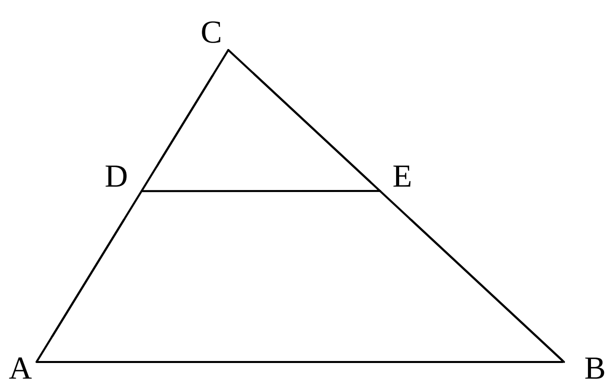
1. Given: ΔACE ~ ΔTOE
   AE = (x + 5) in
   ET = (x – 2) in
   CO = 12 in
   CE = 8 in

   Find the value of *x* in the figure.



2. Given: ΔABC with $\overline{DE} \parallel \overline{AB}$
   DC = 4 un
   AD = (x + 3) un
   CE = 5 un
   BE = 7 un
   DE = 6 un

   Find the value of *x* and AB.



In addition to the AA Similarity Postulate, we have two other ways to prove two triangles are similar using the SSS Similarity Theorem and the SAS Similarity Theorem as stated below.

SSS Similarity Theorem
Two triangles are similar if the measures of the sides of one triangle are proportional to the measures of the corresponding sides of the other triangle.

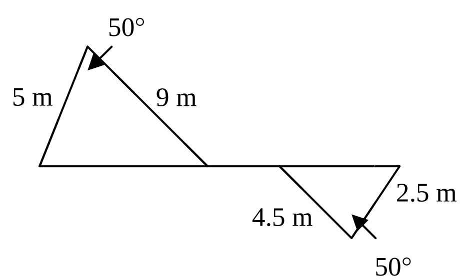
SAS Similarity Theorem
Two triangles are similar if the measures of two sides of one triangle have a proportional relationship with the measures of two corresponding sides of the other triangle and the corresponding included angles are congruent.

We will apply these theorems in the following examples and exercises.

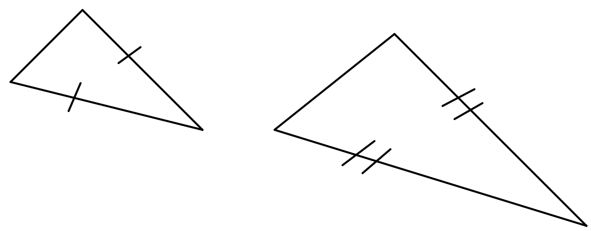
Examples

1. Determine which pair(s) of triangles below are similar and justify each answer.

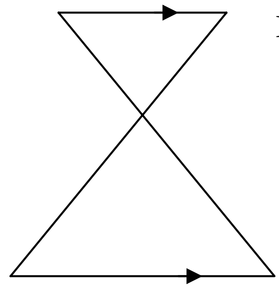
a)



b)



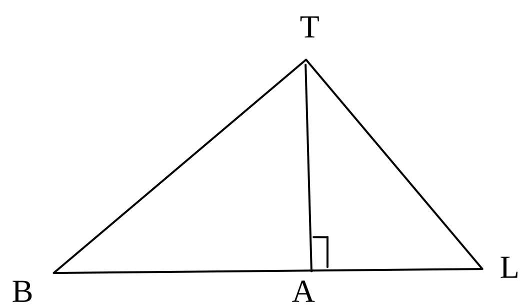
c)



Note: The arrows on the sides indicate that these sides are parallel.

2. Given: Right triangle BLT with right angle ∠BTL
   Altitude $\overline{TA}$
   AL = 6 cm
   TL = 10 cm

   Find: BL



Solutions:

1. a) The triangles are similar by the SAS similarity theorem. The measures of two sides of one triangle are in proportion with the measures of the corresponding sides of the other triangle as shown below.

$$\frac{5}{2.5} = \frac{9}{4.5} = \frac{2}{1}$$

The included angle between the sides of one triangle is congruent to the included angle between the corresponding sides of the other triangle.

b) These triangles are not similar as shown. More information is needed to show the two triangles are similar. We need another pair of sides with measures in the same ratio as the given corresponding sides or we need an included angle in each with the same measure.

c) These triangles are similar by the AA similarity postulate. We have congruent vertical angles in each of the triangles and congruent alternate interior angles.

2. Right triangle BLT is similar to right triangle TLA as follows:

A $\angle$BTL ≅ $\angle$TAL (congruent right angles)

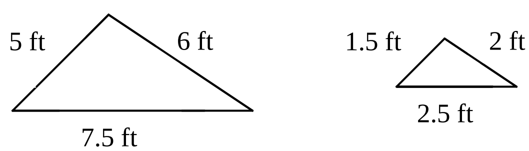
A $\angle$TLB ≅ $\angle$TLB (reflexive property)

The measures of corresponding sides of similar triangles are in proportion. $\triangle BLT \cong \triangle TLA$. So TL/AL = BL/TL. i.e. BL= 100/6 = 16.67.

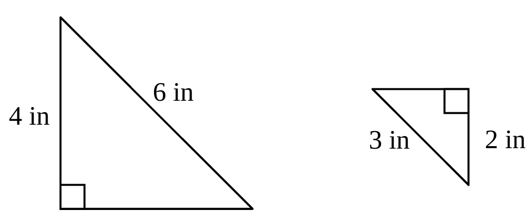
Exercises

1. Determine which pair(s) of triangles below are similar. State a justification for each answer.

a)



b)



2. Given: Rhombus ABCD
DC = 12 m
AT = 9 m
MC = 8 m

Find: TM