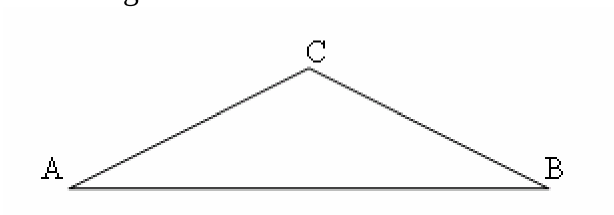
Triangles

<u>Triangle</u>

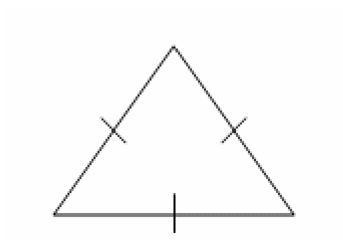
A <u>triangle</u> is a closed figure in a plane consisting of three segments called sides. Any two sides intersect in exactly one point called a vertex.

A triangle is named using the capital letters assigned to its vertices in a clockwise or counterclockwise direction. For example, the triangle below can be named triangle ABC in a counterclockwise direction starting with the vertex A.

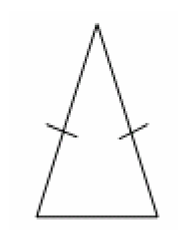


a. What are other names for triangle ABC?

A triangle can be classified according to its sides, angles, or a combination of both. If a triangle has three congruent sides, it is called an <u>equilateral triangle</u> as shown below.



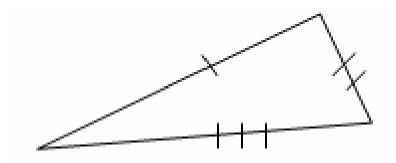
A triangle with at least two sides congruent is called an <u>isosceles triangle</u> as shown below.



b. Are all equilateral triangles isosceles? Why or why not?
c. Are some isosceles triangles equilateral? Explain.
Answers to questions a-c:
a. Triangle (Δ) ACB, ΔBAC, ΔBCA, ΔCAB, ΔCBA

b. All equilateral triangles are also isosceles triangles since every equilateral triangle has at least two of its sides congruent.
c. Some isosceles triangles can be equilateral if all three sides are congruent.

A triangle with no two of its sides congruent is called a <u>scalene triangle</u> and is shown below.



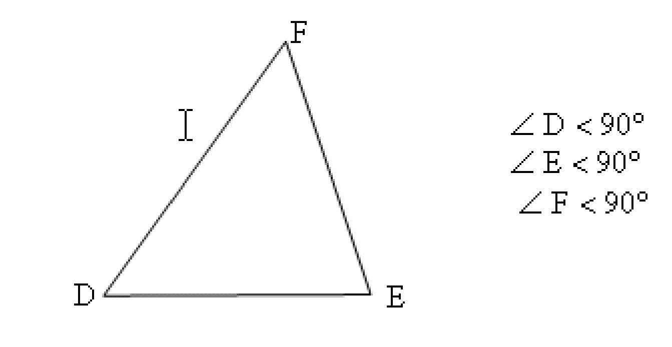
Classification of Triangles by Sides

<u>Equilateral triangle</u>: a triangle with three congruent sides

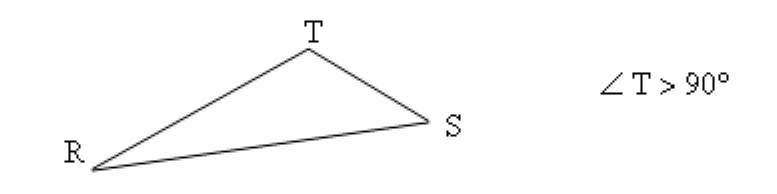
<u>Isosceles triangle</u>**:** a triangle with at least two sides congruent

<u>Scalene triangle</u>**:** a triangle with no two sides congruent

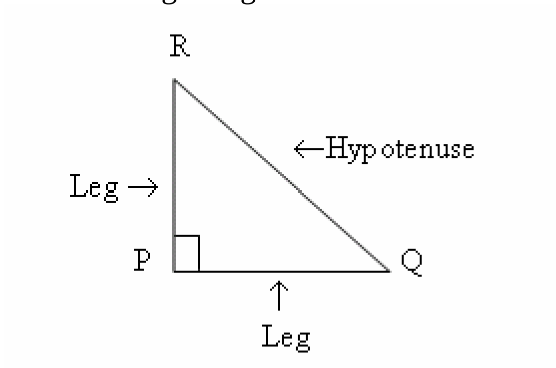
Another way to classify triangles is according to their angles. A triangle with three acute angles can be classified as an <u>acute triangle.</u>



A triangle with one obtuse angle can be classified as <u>obtuse triangle</u>.



A right triangle is a triangle with one right angle.



Segments PQ and RP are called the legs of the right triangle and segment RQ is called the hypotenuse. The legs form the right angle ∠ RPQ. The side opposite the right angle is hypotenuse RQ.

Classification of Triangles by Angles

Acute triangle: a triangle with three acute angles

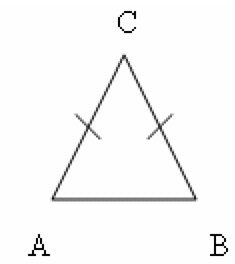
Obtuse triangle: a triangle with one obtuse angle

Right triangle: a triangle with one right angle

Exercises

True or False**:** Give a reason or counterexample to justify your response.

1. An equilateral triangle is always acute.
2. An obtuse triangle can also be isosceles.
3. The acute angles of a right triangle are complementary.
4. Use the figure below and find the value of *x* for each of the following.

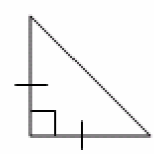


a) AC= (x^2-2x+4) and BC= $(x^2+3x-11)$.
b) BC= 17+3x and AC= x+25
c) AC= x^2-6x and BC= x-12

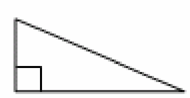
5. Given ΔABC with vertices A(1,5), B(5,5), and C(5,1)
   a) graph ΔABC in the coordinate plane.
   b) classify this triangle by its sides and angles.

Triangles can also be classified by using a combination of angle and side descriptors.

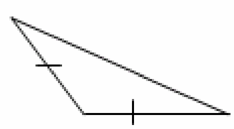
Examples
Right isosceles triangle



Right scalene triangle



Obtuse isosceles triangle



Exercises

Complete each statement below with *always*, *sometimes*, or *never* and give a justification for your answer.

1. A scalene triangle is __________ an acute triangle.
2. A right triangle is ___________ an obtuse triangle.
3. An isosceles triangle is__________ a right triangle.
4. An equilateral triangle is ___________ an isosceles triangle.
5. The acute angles of a right triangle are_________ supplementary.
6. A right isosceles triangle is __________ equilateral.

Exploration

Using linguine, snap off the ends to make segments 3, 5, 6, and 9 inches long.

1. Determine which sets of three lengths will make a triangle.
2. Which sets of three segments did not form a closed figure in the plane?
3. What do the sets that form a triangle have in common?

<u>Solution</u>*:*

1. A triangle can be formed using the following sets of lengths:
   3, 5, 6   5, 6, 9

2. The set consisting of 3, 6, and 9 did not form a triangle. 3+6=9

3. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

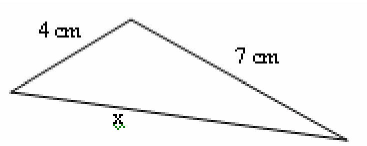
This exploration leads to the following theorem:

Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

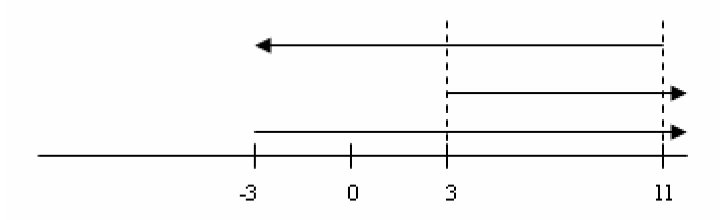
Example

1. Two sides of a triangle have lengths of 4 cm and 7 cm. What are the possible lengths for the third side?



4 cm + 7 cm > x   and   4 cm + x > 7 cm   and   7 cm + x > 4 cm
11 cm > x   and   x > 3 cm   and   x > -3 cm

The intersection of these inequalities can be represented graphically as the intersection of three rays with open endpoints as shown below.



All possible lengths of the third side are represented by the inequality
11cm > x > 3 cm.

Exercises

1. The lengths of three segments are given. Determine if these segments can be used to form a triangle.

   a) 11 cm, 15 cm, and 23 cm
   b) 7.5 in, 8.3 in, and 4.2 in

2. The lengths of two sides of ΔABC are given as A=12 ft and BC=17 ft. What are the possible lengths of the third side AC?

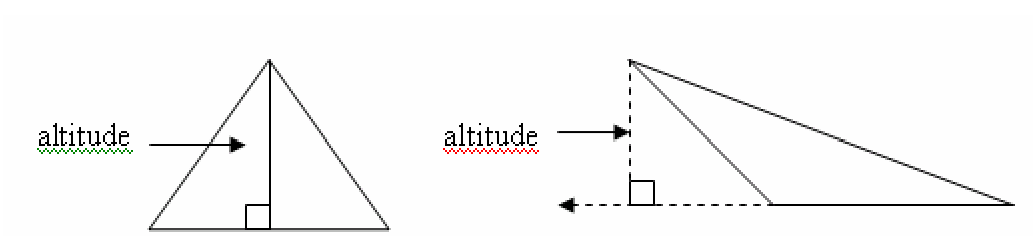
3. ΔDEF has side lengths as follows: DF=(x+1) m , DE=(3x-4) m , and EF= (x+7) m. What are the possible values of *x* ?

Segments of Triangles

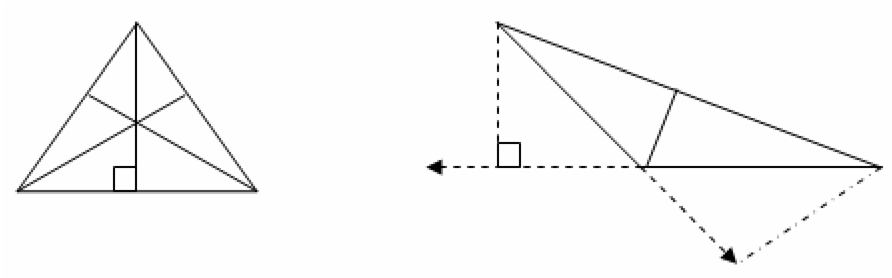
We will discuss three segments in a triangle: altitudes, medians, angle bisectors

Definition

An altitude of a triangle is the segment drawn from a vertex perpendicular to the opposite side or extension of that side.



Every triangle has three altitudes as shown in the figures below.



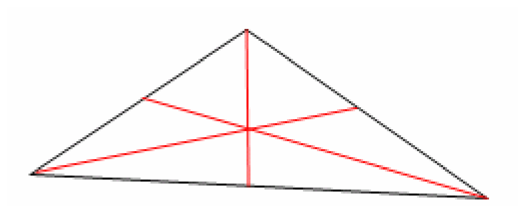
Exploration

In the previous drawings, it seems that the altitudes intersect in a common point. Investigate this idea by using paper folding with patty paper.

a) Draw a large triangle on a sheet of patty paper.
b) Cut out the triangle along its sides.
c) Fold the altitudes of this triangle.
d) The common point of intersection of these altitudes is called the <u>orthocenter.</u>

> **Definition**
>
> A <u>median of a triangle</u> is a segment having one endpoint at a vertex of a triangle and the other endpoint at the midpoint of the opposite side.

A triangle also has three medians as shown in the diagram below.



Exploration

The medians in the drawing also seem to meet in a common point. Use patty paper and paper folding to verify this idea.

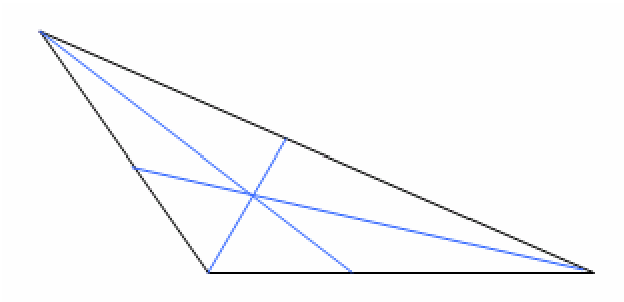
a) Draw a large triangle on a sheet of patty paper.
b) Cut out the triangle along its sides.

c) Crease each segment in the middle after matching its endpoints by folding the paper. This point that divides each segment into two congruent segments is called a midpoint.
d) Make another fold connecting the midpoint of a side with the opposite vertex to form the median. Repeat this process for the other two sides.
e) The point where all three medians intersect is called the centroid or center of mass.

Definition

An angle bisector of a triangle is the segment that bisects an angle of a triangle with one endpoint at the vertex of the angle bisected and the other endpoint on the opposite side of the triangle.

Every triangle has three angle bisectors as shown in the figure below.



Exploration

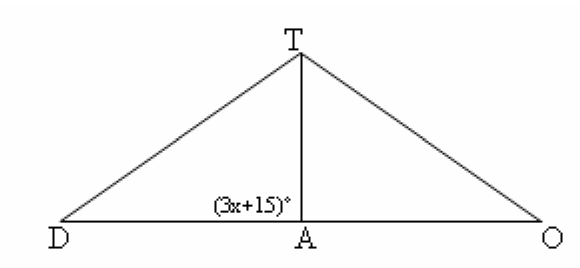
We have medians and altitudes intersecting in a common point and it seems that the angle bisectors also have a common point of intersection. Use paper folding with patty paper to investigate this idea.

a) Begin by drawing a large triangle on a sheet of patty paper.
b) Use scissors to cut out the triangle along its sides.
c) Hold an angle at its vertex and fold so that the sides meet along a line that includes the vertex. Continue this process and fold the other angle bisectors.
d) The common point of intersection of these angle bisectors is called the incenter, the center of the inscribed circle in the triangle.

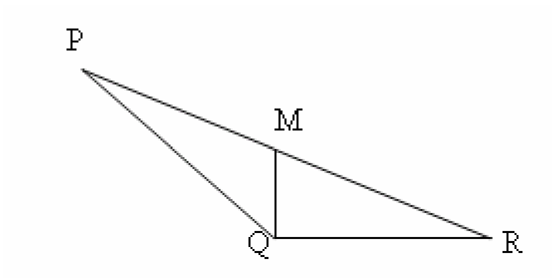
It has been shown that the altitudes, medians, and angle bisectors each have a common point of intersection called a point of concurrency.

Examples

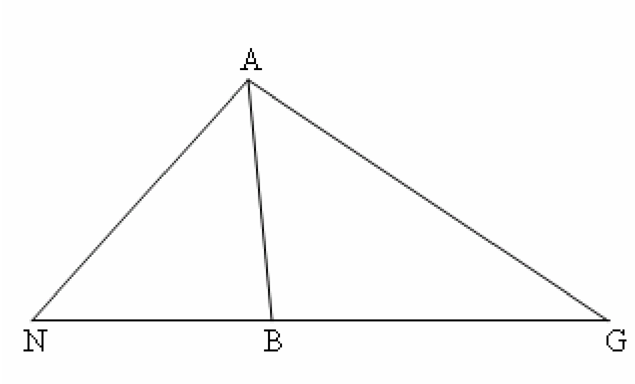
1. Given: ΔDOT as shown
   Find the value of *x* so that $\overline{AT}$ is an altitude.



2. Given: ΔPQR as shown
   PM = (3x- 8) in
   MR = (x + 5) in
   Find the value of *x* so that $\overline{RM}$ is a median.



3. Given: ΔANG as shown below
   m∠NAB = (5x – 4)
   m∠GAB = (3x + 10)
   Find: x so that $\overline{AB}$ is the angle bisector of ∠NAG



Solutions:

1. $\overline{TA} \perp \overline{DO}$ (Definition of an altitude)
   $\angle$TAD is a right angle (⊥s form right angles)
   m$\angle$TAD = 90 (Right $\angle$s have a measure of 90°.)
   3x + 15 = 90 (Substitution)
   3x = 75 (Subtraction property of equality)
   x = 25 (Division/multiplication property of equality)

2. PM = MR (Definition of a median)
   3x – 8 = x + 5 (Substitution)
   2x = 13 (Addition property of equality)
   x = 6.5 (Division/multiplication property of equality)

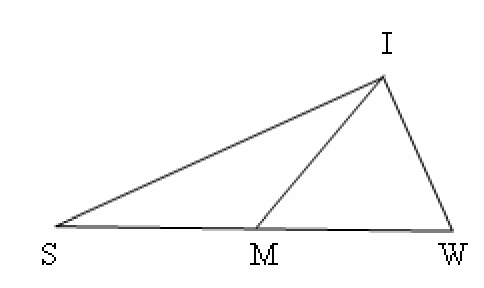
3. m$\angle$NAB = m$\angle$GAB (Definition of an angle bisector)
   5x – 4 = 3x + 10 (Substitution)
   2x = 14 (Addition property of equality)
   x = 7 (Division/multiplication property of equality)

Exercises

1. Given: ΔSWI

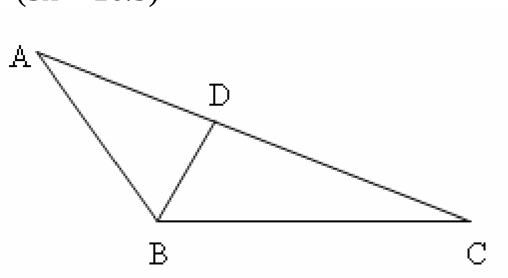
   SM = $(\frac{1}{2}x + 3)$ cm

   MW = $(\frac{2}{3}x - 1)$ cm



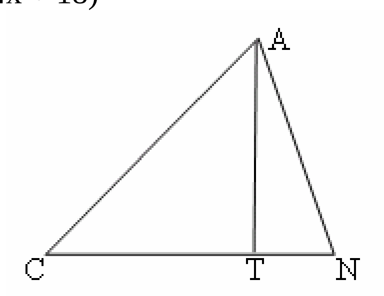
Find: x so that $\overline{IM}$ is a median

2. Given: ΔABC
   m∠ABD = (5x – 7.5)
   m∠CBD = (3x + 16.5)



Find: x so that $\overline{BD}$ is an angle bisector

3. Given: ΔCAN
   m∠ATN = (4x + 18)



Find: x so that $\overline{AT}$ is an altitude

Congruent Triangles

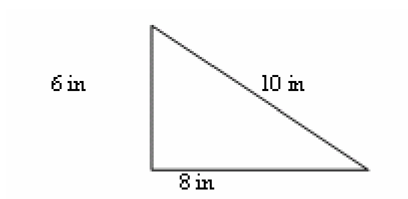
Exploration

Cut pieces of linguine into lengths of 6 in, 8 in, and 10 in.

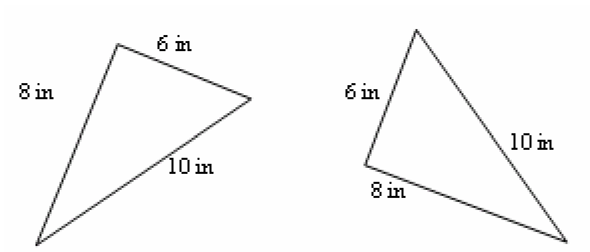
1. Use the pieces of linguine to form a triangle.
2. Is it possible to form a different triangle using these lengths? Explain.
3. How do these triangles compare?

Solutions:

1. The pieces of linguine can be used to form the following triangle.



2. It is possible to form triangles with different orientations in the plane as shown below.



3. The triangles have the same size and shape as the original triangle shown.

Exploration

Use a piece of tracing or patty paper to trace the triangles in solution 2. Use rotations and translations to match corresponding sides .

1. How do the corresponding angles compare?
2. How many parts of one triangle match with corresponding parts of another triangle having the same size and shape?
3. What is the relationship between corresponding sides and corresponding angles in the set of triangles?

Solutions:

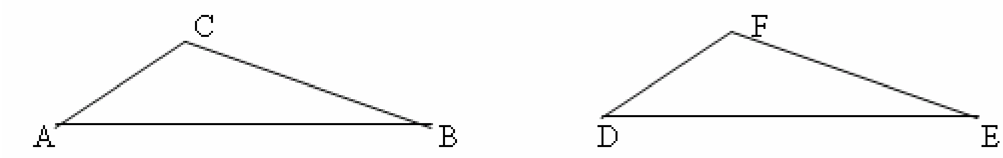
1. The corresponding angles have the same measure.
2. Three sides and three angles of one triangle match with three corresponding sides and three corresponding angles of another triangle.
3. Corresponding sides are opposite corresponding angles. The triangles in solution 1 and solution 2 are said to be congruent.

Congruent Triangles

Two triangles are congruent if and only if their corresponding sides and their corresponding angles are congruent.

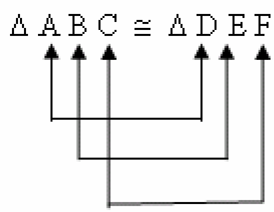
Examples

1. Given triangle ABC is congruent to triangle DEF. Identify the corresponding parts in the two triangles.



Another way to state that triangle ABC is congruent to triangle DEF is by using the following notation: $\Delta ABC \cong \Delta DEF$

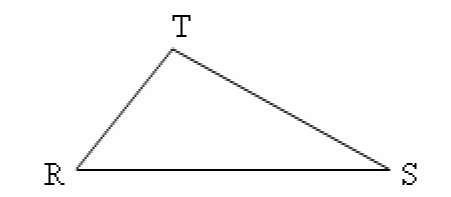
The corresponding sides and corresponding angles can be identified by matching the corresponding vertices of the two triangles as shown below.



$\overline{AB} \cong \overline{DE}$	$\angle ABC \cong \angle DEF$
$\overline{BC} \cong \overline{EF}$	$\angle ACB \cong \angle DFE$
$\overline{AC} \cong \overline{DF}$	$\angle CAB \cong \angle FDE$

The corresponding sides and corresponding angles of two congruent triangles are referred to as corresponding parts of congruent triangles. We often write CPCTC for "**C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent".

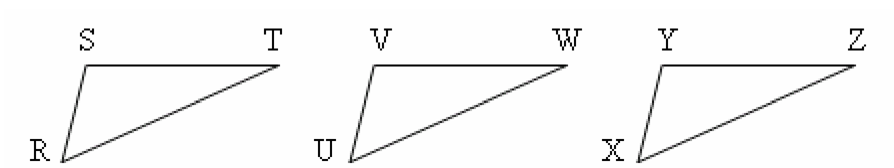
2. Show that the congruence of triangles is reflexive.



Given: ΔRST
Show $\Delta RST \cong \Delta RST$

We know that $\overline{RT} \cong \overline{RT}$, $\overline{RS} \cong \overline{RS}$, and $\overline{TS} \cong \overline{TS}$ by the reflexive property of equality. $\angle R \cong \angle R$, $\angle S \cong \angle S$, and $\angle T \cong \angle T$ by the reflexive property of equality. We now have three sides and three angles of ΔRST congruent to the three corresponding sides and corresponding angles of ΔRST. By the definition of congruent triangles, $\Delta RST \cong \Delta RST$.

3. Show that the congruence of triangles is transitive.



Given: $\Delta RST \cong \Delta UVW$, $\Delta UVW \cong \Delta XYZ$
Show $\Delta RST \cong \Delta XYZ$

We know that $\overline{RS} \cong \overline{UV}$, $\overline{RT} \cong \overline{UW}$, $\overline{ST} \cong \overline{VW}$, $\angle R \cong \angle U$, $\angle S \cong \angle V$, and $\angle T \cong \angle W$ by CPCTC given $\Delta RST \cong \Delta UVW$. Since $\Delta UVW \cong \Delta XYZ$, we have $\overline{UV} \cong \overline{XY}$, $\overline{UW} \cong \overline{XZ}$, $\overline{VW} \cong \overline{YZ}$, $\angle U \cong \angle X$, $\angle V \cong \angle Y$, and $\angle W \cong \angle Z$ by CPCTC. We know that congruent sides have equal measures and by the transitive property of equality we have the following: $RS = XY$, $RT = XZ$, and $ST = YZ$
Using a similar justification, the following is true for corresponding angles:
$m\angle R = m\angle X$, $m\angle S = m\angle Y$, and $m\angle T = m\angle Z$
$\Delta RST \cong \Delta XYZ$ by definition of congruent triangles.

4. Congruence of triangles is also symmetric. This justification is left as an optional exercise.

Exercises

1. Given ΔMNO ≅ ΔPQR, identify the corresponding parts.
2. Given: ΔMNO ≅ ΔPQR, MN = (5x-11), PQ = (3x-2)
   Find: x and MN
3. Given: ΔKLM ≅ ΔNOP, m∠M = (5x + 23), and m∠P = (3x + 40)
   Find: x and m∠P
4. Given: ΔABC with vertices A(-3,2), B(2,6), and C(-3,6) and
   ΔEFG with vertices E(2,2), F(7,-2), and G(2,-2)
   a) Find the lengths of each side and determine if the triangles are congruent.
   b) If the triangles are congruent, write a congruence statement for the two triangles. If the triangles are not congruent, justify your conclusion.

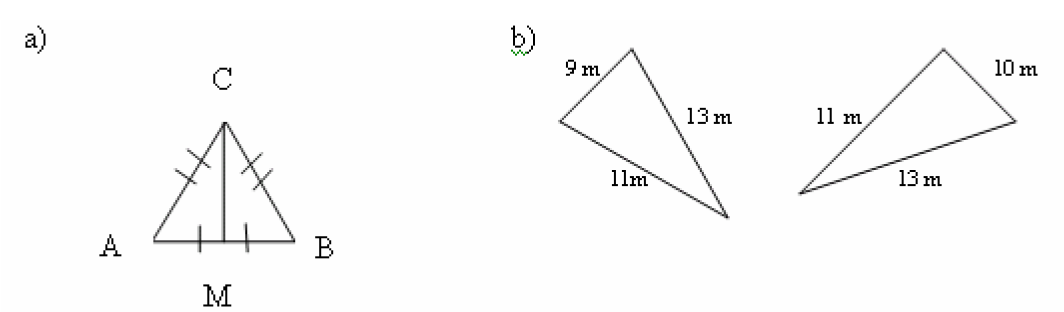
Exploration
By the definition of congruent triangles, two triangles are congruent if and only if the six parts of one triangle are congruent to the six corresponding parts of a second triangle. Is it possible for two triangles to be congruent using only some of the corresponding parts? If so, which corresponding parts are sufficient to show two triangles congruent?

Do problem 1 in the activity ***Look* Alikes** using Cabri Junior™ on a TI-83+ graphing calculator. In this exploration, a triangle will be constructed using three sides of a given triangle. The results of this exploration leads us to the SSS congruence postulate for proving two triangles congruent.

> **SSS Congruence Postulate**
>
> Two triangles are congruent if and only if three sides of one triangle are congruent to three sides of a second triangle.

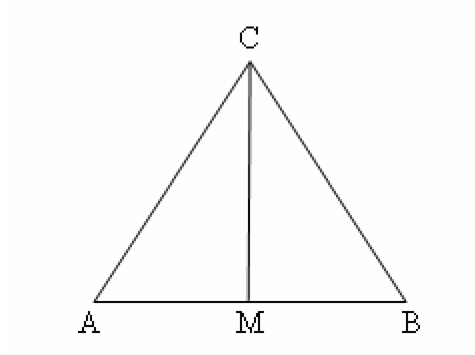
Examples

1. Determine which of the following represents a pair of congruent triangles.



2. Given: ΔACB is isosceles , point M is the midpoint of $\overline{AB}$

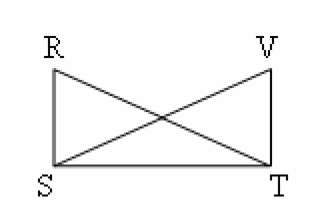
Show $\Delta AMC \cong \Delta BMC$.



$\overline{AC} \cong \overline{CB}$ by the definition of an isosceles triangle. $\overline{AM} \cong \overline{MB}$ since M is the midpoint of $\overline{AB}$. $\overline{CM} \cong \overline{CM}$ by the reflexive property of equality. Now we have $\Delta AMC \cong \Delta BMC$ by the SSS congruency postulate.

3. Given: $\overline{RS} \cong \overline{VT}$, $\overline{RT} \cong \overline{SV}$

Prove: $\angle SRT \cong \angle TVS$



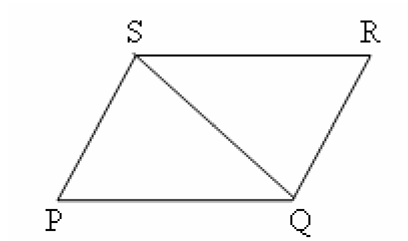
Statements	Reasons
1. $\overline{RS} \cong \overline{VT}$ $\overline{RT} \cong \overline{SV}$	1. Given
2. $\overline{ST} \cong \overline{ST}$	2. Reflexive property of equality
3. $\Delta SRT \cong \Delta TVS$	3. SSS congruence postulate
4. $\angle SRT \cong \angle TVS$	4. CPCTC

Exercises

1. Given: $\Delta MNO \cong \Delta PQR$
   Determine whether each statement is true or false and give a justification for your answer.
   a) $\overline{MN} \cong \overline{PQ}$
   b) $\angle MNO \cong \angle PRQ$
   c) $m\angle OMN = m\angle QPR$

2. Given: $\overline{PQ} \cong \overline{SR}$
   $\overline{PS} \cong \overline{QR}$

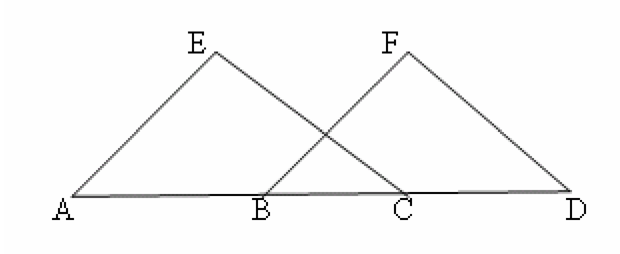
   Prove: $\Delta PQS \cong \Delta RSQ$



Complete the proof with an appropriate statement or reason.

Statements	Reasons
1. $\overline{PQ} \cong \overline{SR}$ $\overline{PS} \cong \overline{QR}$	1. ____________________________
2. ______________________________	2. Transitive property of equality
3. $\Delta PQS \cong \Delta RSQ$	3. ____________________________

3. Given: $\overline{AE} \cong \overline{BF}$
$\overline{EC} \cong \overline{DF}$
$\overline{AB} \cong \overline{CD}$



Prove: ∠AEC ≅ ∠BFD

Proof:

Statements	Reasons
1. $\overline{AE} \cong \overline{BF}$, $\overline{EC} \cong \overline{DF}$, $\overline{AB} \cong \overline{CD}$	1.________________________
2. AE = BF, EC = DF, AB = CD	2. ________________________
3. ________________________	3. Reflexive property
4. AB + BC = CD + BC	4. ________________________
5. AB + BC = AC, CD + BC = BD	5. ________________________
6. ________________________	6. Substitution
7. ________________________	7. SSS congruence postulate
8. ∠AEC ≅ ∠BFD	8. ________________________

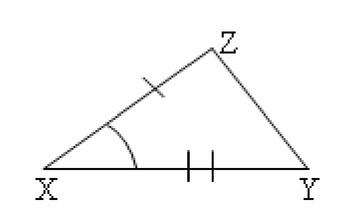
4. Given: ΔRST with vertices R(-3,8), S(2,5), and T(2,8)
ΔMNP with vertices M(10,0), N(5,3), and P(5,0)

Show $\Delta RST \cong \Delta MNP$.

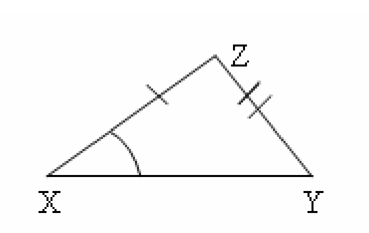
Exploration

Refer to the activity ***Look Alikes*** in the Geometry Module. Use a TI-83+ graphing calculator with Cabri Junior™ and do problem 2. In this problem, a triangle will be constructed using two sides and the included angle of a given triangle. Side lengths and angle measures of the constructed triangle will be compared to the corresponding side lengths and angle measures of the original triangle.

This activity refers to *two sides and the included angle* of a triangle. In the diagram below, the sides $\overline{XZ}$ and $\overline{XY}$ of ΔXYZ are included in the sides of $\angle ZXY$



In the diagram below, the given sides (marked) are not included in the sides of $\angle ZXY$. This diagram does not represent *two sides and the included angle* of a triangle.



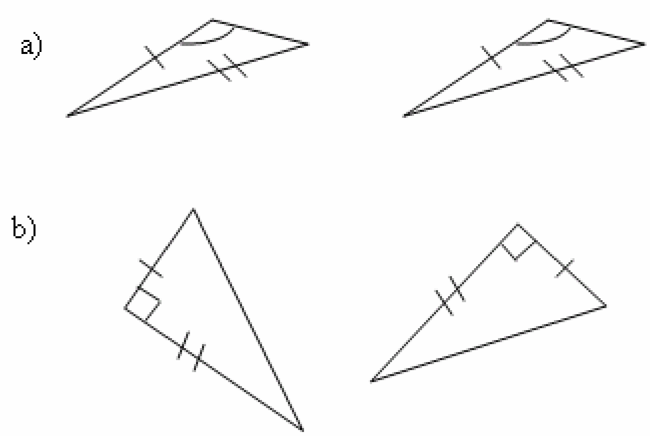
The results of this exploration lead us to the following congruence postulate for triangles.

SAS Congruence Postulate

Two triangles are congruent if and only if two sides and the included angle of one triangle are congruent to two sides and the included angle of the second triangle.

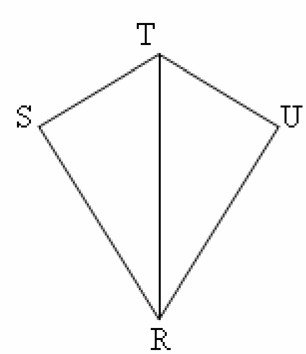
Examples

1. Which of the following represents a pair of congruent triangles? State the congruence postulate that applies and write a justification.

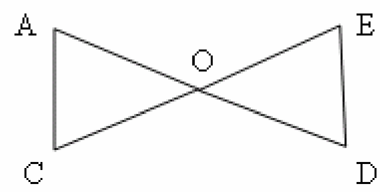


c) Given: $\overline{TR}$ bisects $\angle STU$

$\overline{ST} \cong \overline{TU}$



d) Given: $\overline{AD}$ and $\overline{CE}$ bisect each other at point O
$\triangle AOC \cong \triangle DOE$



<u>Solutions:</u>

1. a) The triangles do not represent a pair of congruent triangles since the angle shown is not the included angle for the two sides given.

   b) The pair of triangles shown are congruent by the SAS congruence postulate since the sides marked are included in the sides of the right angle.

   c) ΔRST ≅ ΔRUT by the SAS congruence postulate as follows:

S	$\overline{ST} \cong \overline{TU}$	These corresponding sides are given.
A	∠STR ≅ ∠RUT	An angle bisector divides an angle into two congruent angles.
S	$\overline{RT} \cong \overline{RT}$	Reflexive property

   d) ΔAOC ≅ ΔDOE by the SAS and SSS congruence postulates.

   Using the SAS congruence postulate, we have the following justification:

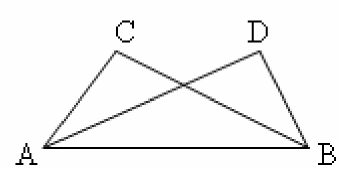
S	$\overline{CO} \cong \overline{EO}$	A bisector of a segment divides it into two congruent segments.
A	∠AOC ≅ ∠DOE	Vertical angles are congruent.
S	$\overline{AO} \cong \overline{DO}$	A bisector of a segment divides it into two congruent segments.

   Using the SSS congruence postulate, we have the following justification:

S	$\overline{AC} \cong \overline{DE}$	Given
S	$\overline{CO} \cong \overline{EO}$	A bisector of a segment divides it into two congruent segments.
S	$\overline{AO} \cong \overline{DO}$	A bisector of a segment divides it into two congruent segments.

Examples

2. Given: $\overline{AC} \cong \overline{BD}$
   $\angle CAB \cong \angle DBA$



Prove: $\overline{AD} \cong \overline{BC}$

Complete the following proof.

Statements	Reasons
1. $\overline{AC} \cong \overline{BD}$ $\angle CAB \cong \angle DBA$	1. ______________________
2. ______________________	2. Reflexive property
3. ______________________	3. SAS congruence postulate
4. $\overline{AD} \cong \overline{BC}$	4. ______________________

3. Given: ΔABC with vertices A(-2,6), B(1,2), and C(1,8)
   ΔDFE with vertices D(7,4), F(4,0), and E(4,6)
   $\angle BAC = \angle FDE = 86.8°$

   a) Graph the coordinates of the vertices of each triangle in the coordinate plane.

b) Determine if $\Delta ABC \cong \Delta DFE$ and state the congruence postulate used.

4. Given: $\Delta KAR \cong \Delta GTO$ with KA = (3x-10) , GT = (x+2), $m\angle KAR = (5y-40)$ and $m\angle GTO = (3y-8)$

Find: x, y and KA

Solutions:

2.

Statements	Reasons
1. $\overline{AC} \cong \overline{BD}$ $\angle CAB \cong \angle DBA$	1. Given
2. $\overline{AB} \cong \overline{AB}$	2. Reflexive property
3. $\Delta CAB \cong \Delta DBA$	3. SAS congruence postulate
4. $\overline{AD} \cong \overline{BC}$	4. CPCTC

3. b) The distance formula can be used to determine the distance between two points with coordinates (x, y) and (x_1,y_1) in the coordinate plane:

$$d = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

Using the distance formula above, the following lengths of sides can be determined.

$$AC = \sqrt{(-2-1)^2 + (6-8)^2} = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$
$$AB = \sqrt{(-2-1)^2 + (6-2)^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$
$$ED = \sqrt{(4-7)^2 + (6-4)^2} = \sqrt{(-3)^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$$

$DF = \sqrt{(7-4)^2 + (4-0)^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

We now have $\overline{AC} \cong \overline{DE}$ and $\overline{AB} \cong \overline{DF}$ since congruent segments have equal measures. $\angle$BAC and $\angle$FDE have the same measure and are also congruent. This gives us two sides and an included angle of ΔABC congruent to two sides and an included angle of ΔDFE. Therefore, $\Delta ABC \cong \Delta DFE$ by the SAS congruence postulate.

4. We have KA = GT from the congruence statement. This allows us to write and solve the following equation:

$$3x - 10 = x + 2$$
$$2x = 12$$
$$x = 6$$

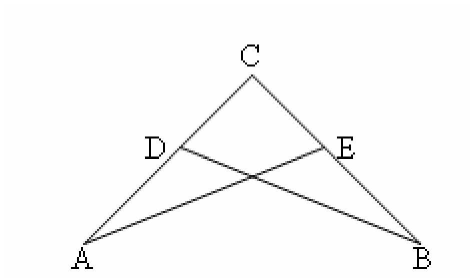
To find KA, substitute the value 6 for *x* in the equation KA = (3x-10) and get KA = (3•6-10) = 8.
Since corresponding angles of congruent triangles are congruent, m$\angle$KAR = m$\angle$GTO. By substituting the algebraic expressions for these measures, we have the following equation: 5y-40 = 3y-8

Solve this equation for *y* :

$$2y = 32$$
$$y = 16$$

Exercises

1. Given: $\overline{AC} \cong \overline{BC}$
   $\overline{CD} \cong \overline{CE}$



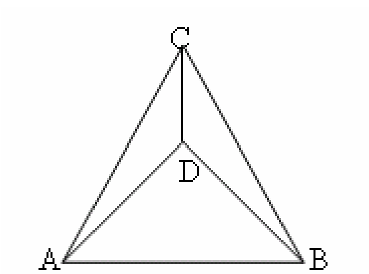
Prove: $\angle A \cong \angle B$

Complete the following proof.

Statements	Reasons
1. ______________________________ ______________________________	1. Given
2. ______________________________	2. Reflexive property
3. $\Delta ACE \cong \Delta BCD$	3. ______________________________
4. ______________________________	4. CPCTC

2. Given: ΔACB is isosceles with base $\overline{AB}$
   ray CD bisects $\angle ACB$
   Show ΔADB is isosceles

   Write a justification explaining why ΔADB is isosceles. This justification may be written in the two-column format like the proof in exercise 1 or in a paragraph format.



3. Given: ΔPQR with vertices P(-3,1), Q(1,1), and R(1,6)
   ΔSTU with vertices S(8,2), T(4,2), and U(4,7)

   a) Graph the triangles in the coordinate plane.
   b) Determine if ΔPQR and ΔSTU are congruent.
   c) Justify your answer.

4. Given: Right triangle ACB with right $\angle ACB$
   Right triangle DEF with right $\angle DEF$
   $\overline{AC} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$

   Prove: $\Delta ACB \cong \Delta DEF$

   This proof leads to the following theorem for proving right triangles congruent.

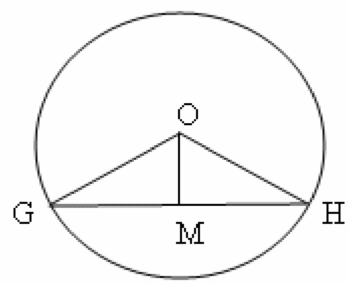
Leg-Leg Theorem

If two legs of one right triangle are congruent to two legs of a second right triangle, then the right triangles are congruent.

5. Given: Circle O with radii $\overline{OG}$ and $\overline{OH}$

   $\overline{OM}$ bisects ∠GOH
   GM = $x^2 + 2x + 5$
   HM = $x^2 + x + 11$
   OM = $3x - 6$

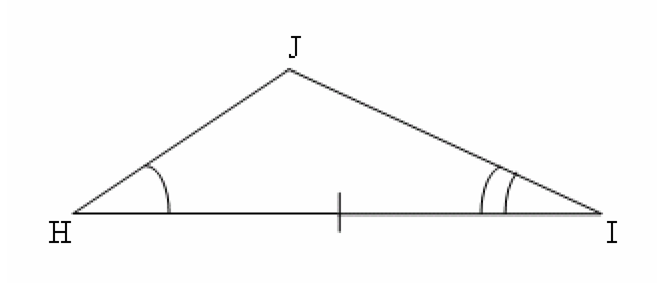
   Find: x, GH, and OM



Exploration

Refer to the activity ***Look Alikes*** in the Geometry Module. Use a TI-83+ graphing calculator with Cabri Junior™ and do problem 3. In this problem, a triangle will be constructed using two angles and the <u>included side</u> of a given triangle. Angle measures and side lengths of the constructed triangle will be compared to the corresponding angle measures and side lengths of the original triangle.

This activity refers to <u>*two* angles *and the included side*</u> of a triangle. In the diagram below, $\overline{HI}$ is the <u>included side</u> between ∠H and ∠I.



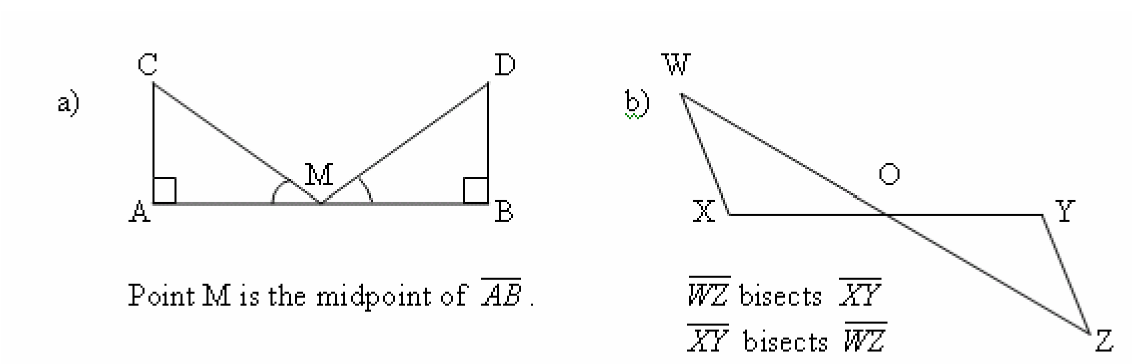
The results of this exploration lead us to the following congruence postulate for triangles.

ASA Congruence Postulate

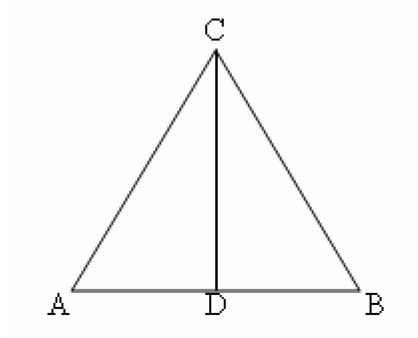
Two triangles are congruent if and only if two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle.

Examples

1. Which of the following pairs of triangles are congruent by the ASA congruence postulate? Justify your reasoning.

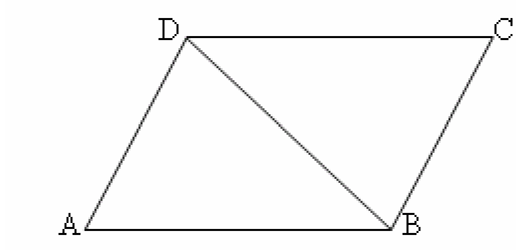


2. Given: $\overline{CD}$ bisects ∠ACB
   $\overline{CD} \perp \overline{AB}$



Prove: ΔACB is isosceles
using a flow-chart proof.

3. Given: $\angle$ADB ≅ $\angle$CBD
$\angle$ABD ≅ $\angle$CDB
m $\angle$A = 3x + 15
m $\angle$C = 8x – 20



Find: x and m $\angle$A

Solutions:

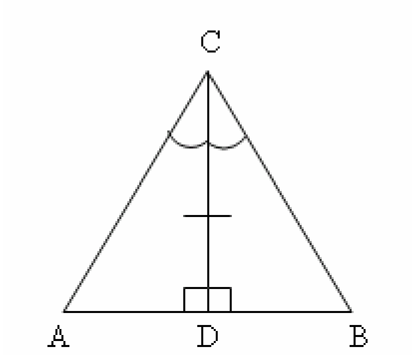
1. a) ΔCAM ≅ ΔDBM by the ASA congruence postulate.
   - **A** $\angle$A ≅ $\angle$B (Right angles are congruent.)
   - **S** $\overline{AM} \cong \overline{MB}$ (A midpoint divides a segment into two congruent segments.)
   - **A** $\angle$AMC ≅ $\angle$BMD (Given)

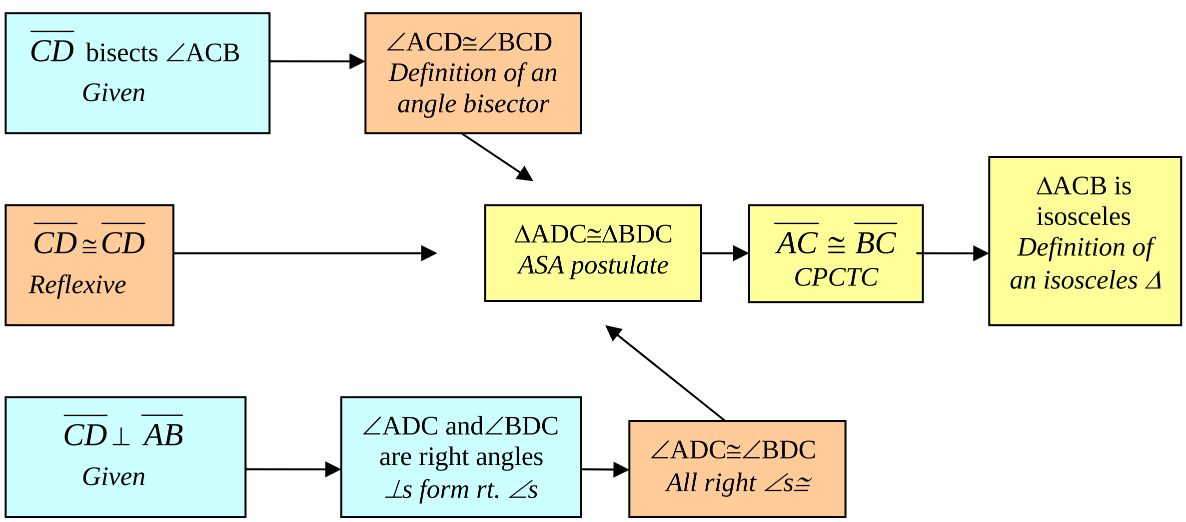
   b) ΔWOX ≅ ΔZOY by the SAS congruence postulate. The ASA congruence postulate does not apply to this problem.
   - **S** $\overline{XO} \cong \overline{YO}$ (A bisector divides a segment into two congruent segments.)

A ∠XOW ≅ ∠YOZ (Vertical angles are congruent.)

S $\overline{WO} \cong \overline{OZ}$ (A bisector divides a segment into two congruent segments.)

2. A flow-chart proof shows the logical development of a proof using statements with supporting justification(s) in a flow-chart format. The statements and reasons are shown in rectangles with arrows indicating how they connect to other information in rectangles. The flow of information should lead to the conclusion. The proof that follows models one format that can be used.





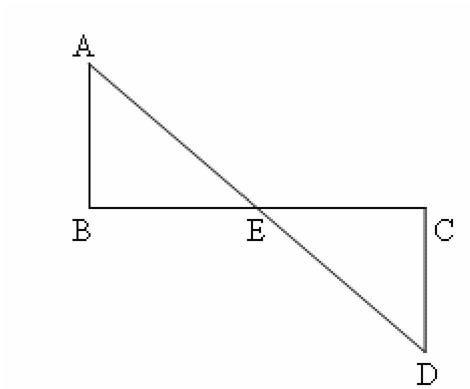
3. ΔADB ≅ ΔCBD by the ASA congruence postulate.

A	∠ADB ≅ ∠CBD	(Given)
S	$\overline{DB} \cong \overline{DB}$	(Reflexive property)
A	∠ABD ≅ ∠CDB	(Given)
	∠A ≅ ∠C	(CPCTC)
	m ∠A = m ∠C	(Congruent angles have equal measures.)
	3x+15 = 8x-20	(Substitution)
	35 = 5x	(Addition and subtraction properties of equality)
	7 = x	(Division property of equality)

m ∠A = 3x + 15 = 3•7 + 15 = 21 + 15 = 36

Exercises

1. Given: $\overline{AD}$ bisects $\overline{BC}$
   $\overline{AB} \perp \overline{BC}$
   $\overline{DC} \perp \overline{BC}$



Prove: $\overline{BC}$ bisects $\overline{AD}$

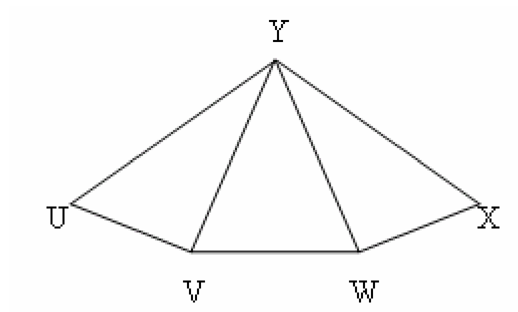
Proof:
Complete the proof below.

Statements	Reasons
1. $\overline{AD}$ bisects $\overline{BC}$ $\overline{AB} \perp \overline{BC}$ $\overline{DC} \perp \overline{BC}$	1. ______________________
2. $\angle B$ and $\angle C$ are right angles	2. ______________________
3. ______________________	3. All right angles are congruent.
4. $\overline{BE} \cong \overline{BC}$	4. ______________________
5. $\angle AEB \cong \angle DEC$	5. ______________________
6. ______________________	6. ASA congruence postulate
7. ______________________	7. CPCTC
8. $\overline{BC}$ bisects $\overline{AD}$	8. ______________________

2. Given: ΔYVW is isosceles with base $\overline{VW}$

$\angle UYV \cong \angle XYW$

$\angle UVY \cong \angle XWY$



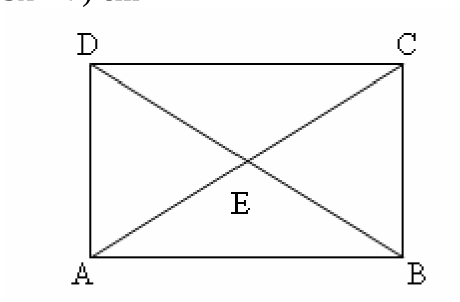
Prove $\angle U \cong \angle X$ using a flow-chart proof.

3. Given: $\angle BAC \cong \angle DCA$

$\overline{AE} \cong \overline{EC}$

DC = $(x^2 + 2x - 1)$ cm

AB = $(x^2 + 5x - 7)$ cm



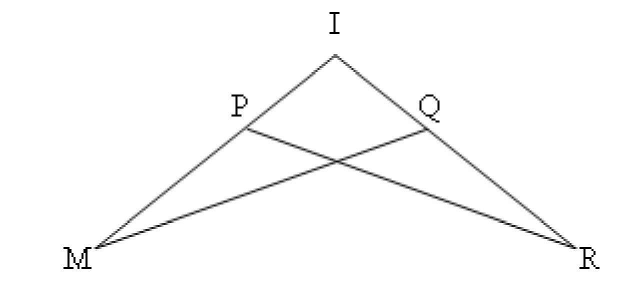
Find: x and AB

4. Given: $\angle M \cong \angle R$

$\overline{MI} \cong \overline{RI}$

m $\angle RPI = (9x - 10)$

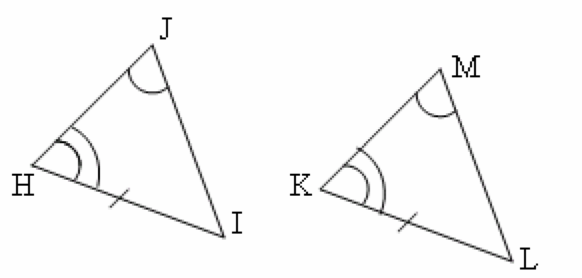
m $\angle MQI = (4x + 45)$



Find: x and m ∠MQI

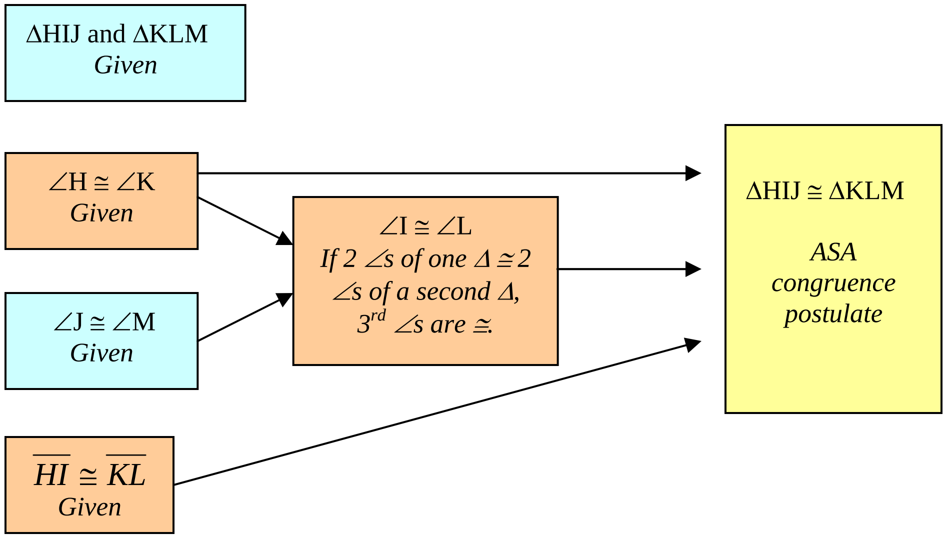
We have explored the SSS, SAS, and ASA congruence postulates and LL theorem (right triangles only). Are there other ways to show two triangles congruent? We will consider the possibility of using <u>any two angles and a side (not included) of one triangle</u> (AAS) congruent to the corresponding parts of another triangle in the following proof.

Given: ΔHIJ and ΔKLM
∠H ≅ ∠K, ∠J ≅ ∠M
$\overline{HI} \cong \overline{KL}$



Prove: ΔHIJ ≅ ΔKLM

Proof:



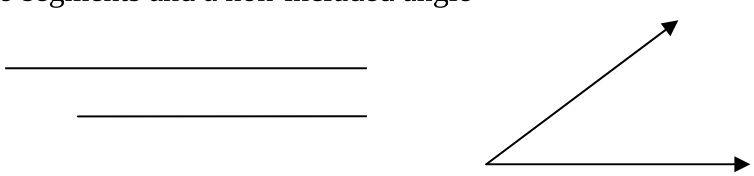
This proof leads to the following theorem for showing two triangles congruent.

> **AAS Congruence Theorem**
>
> If two angles and a side (not included) are congruent to the corresponding parts of a second triangle, then the two triangles are congruent.

Is it possible for two triangles to be congruent using SSA where the angle is not included between the two sides? Make a conjecture: ____
____.
This conjecture will be tested in the following exploration.

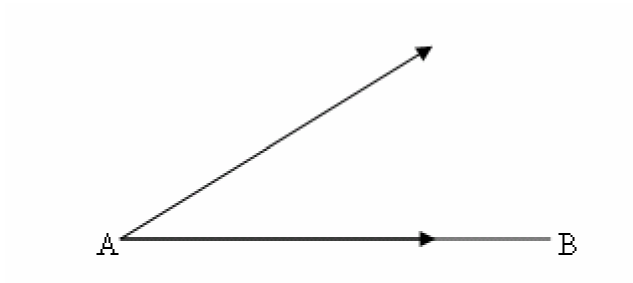
Exploration

Given: Two segments and a non-included angle

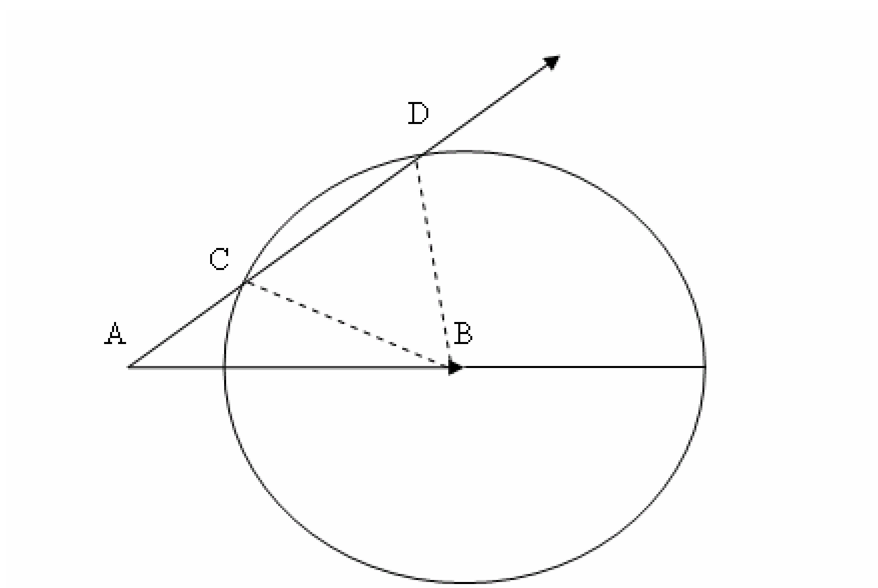


Patty paper, a compass, and straight edge will be needed for this exploration. Follow the procedure outlined below.

a) Trace the longer segment given ($\overline{AB}$) on a sheet of patty paper.
b) Place the patty paper over the given angle and align the vertex of the angle with endpoint A of the longer segment. Align one of the sides of the angle with $\overline{AB}$.
c) Trace the angle and extend the side not aligned with the segment as shown below.



d) Construct circle B with a radius equal to the length of the shorter segment given. (The construction below is not drawn to scale.)



Circle B intersects ray AD in two points, point C and point D. There are two possible triangles that can be constructed with the given ∠CAB, $\overline{AB}$, and the shorter segment represented by the radii $\overline{BC}$ and $\overline{BD}$. ΔABC and ΔABD are the two triangles shown in the figure above. Therefore, it is not possible to construct a unique triangle given two sides and a non-included angle.

Examples

1. Given: $\triangle CED$ is isosceles with base $\overline{CD}$

   $\overline{GB} \perp \overline{AD}$

   $\overline{CF} \perp \overline{GE}$

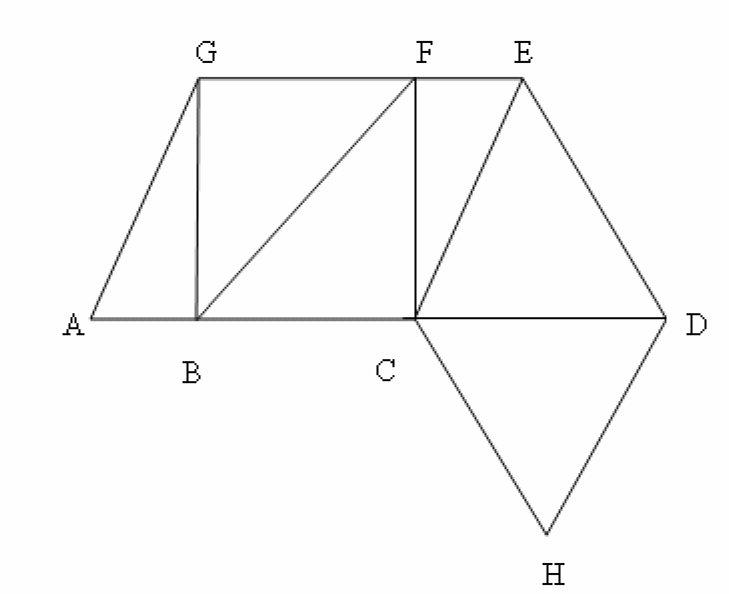
   CH = ED

   CE = AG

   $\angle A \cong \angle FEC$

   $\angle GBF \cong \angle CFB$

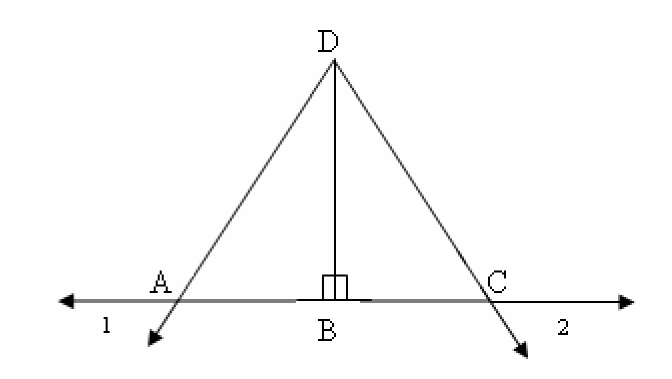
   Identify all pairs of congruent triangles in the figure given and explain the congruence postulate or theorem used.



2. Given: $\angle 1 \cong \angle 2$

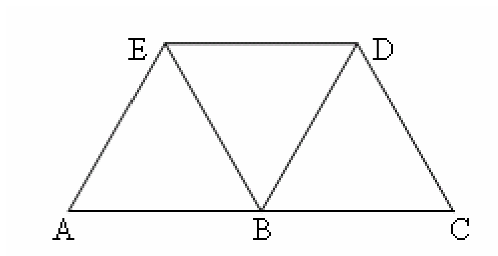
   $\overline{DB} \perp \overline{AC}$

   Prove: $\overline{DB}$ bisects $\angle ADC$



Write a paragraph, two-column, or flow-chart proof.

3. Given: ΔDBE is isosceles with base $\overline{DE}$
$\angle A \cong \angle C$
$\angle ABE \cong \angle CBD$
$AB = (\frac{1}{2}x + 1)$ cm
$BC = (\frac{3}{4}x - 5)$ cm



Find: x, AB, and BC

<u>Solutions:</u>

1. $\Delta ABG \cong \Delta EFC$ by the AAS congruence theorem

A	$\angle A \cong \angle FEC$	(Given)
A	$\angle ABG \cong \angle EFC$	($\perp$s form rt. $\angle$s and all rt. $\angle$s $\cong$)
S	$\overline{AG} \cong \overline{CE}$	(CE=ED=AG; $\cong$ seg. have = meas.)

$\Delta CDE \cong \Delta CDH$ by the SSS congruence postulate

S	$\overline{CH} \cong \overline{DE}$	(Given; $\cong$ seg. have = meas.)
S	$\overline{CE} \cong \overline{DH}$	(CE=ED=CH=DH; $\cong$ seg. have = meas.)
S	$\overline{CD} \cong \overline{CD}$	(Reflexive property)

$\Delta GBF \cong \Delta CFB$ by the SAS congruence postulate

S	$\overline{GB} \cong \overline{CF}$	($\Delta ABG \cong \Delta EFC$; CPCTC)

A $\angle GBF \cong \angle CFB$ (Given)
S $\overline{BF} \cong \overline{BF}$ (Reflexive property)

2. Triangle ABD will be shown congruent to triangle CBD using the AAS congruence theorem in a paragraph-style proof as follows.

   Angle 1 is congruent to $\angle DAB$ and $\angle 2$ is congruent to $\angle DCB$ because vertical angles are congruent. Angle DBA and $\angle DBC$ are right angles since $\overline{DB} \perp \overline{AC}$. It follows that $\angle DBA \cong \angle DBC$ because all right angles are congruent. Side DB is congruent to itself by the reflexive property. We now have $\Delta ADB \cong \Delta CDB$ by the AAS congruence theorem. Angle ADB is congruent to $\angle CDB$ by CPCTC. Therefore, $\overline{DB}$ bisects $\angle ADC$.

3. $\Delta ABE \cong \Delta CBD$ by AAS
   A $\angle A \cong \angle C$ (Given)
   A $\angle ABE \cong \angle CBD$ (Given)
   S $\overline{BE} \cong \overline{BD}$ (An isosceles triangle has 2 congruent sides.)
   $\overline{AB} \cong \overline{BC}$ (CPCTC)
   AB = BC (Congruent segments have equal measures.)
   $\frac{1}{2}x + 1 = \frac{3}{4}x - 5$ (Substitution)

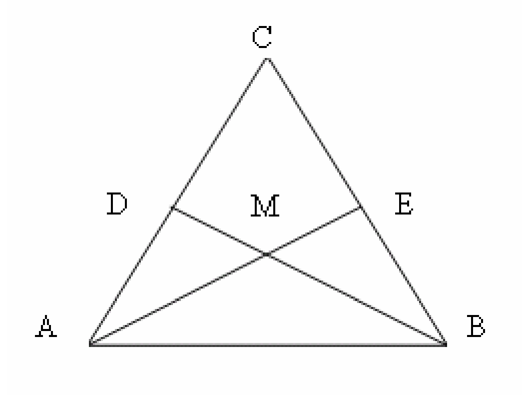
   $6 = \frac{1}{4}x$
   $24 = x$

   $AB = \frac{1}{2} \bullet 24 + 1 = 13$ cm CD = AB = 13 cm

Exercises

1. Given: $\angle BAE \cong \angle DBA$
   $\overline{DB} \cong \overline{AE}$

   a) Identify all pairs of congruent triangles in the given figure.
   b) Explain the congruence postulate or theorem used.



2. Given: $\angle RVS \cong \angle USV$

   $\angle RSV \cong \angle UVS$

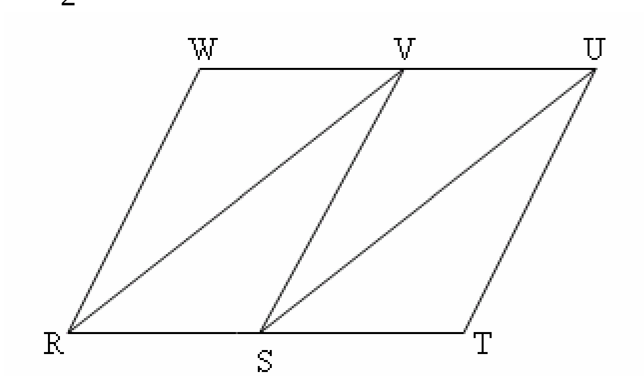
   $\angle WRV \cong \angle TUS$

   $\angle W \cong \angle T$

   $VW = (\frac{2}{3}x + 3)$ in

   $ST = (\frac{5}{6}x - 1)$ in

   $RS = (\frac{1}{2}x + 5)$ in

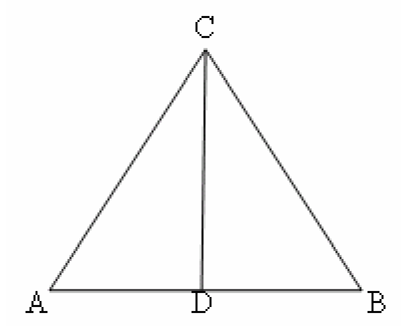


3. Given: Isosceles ΔABC with base $\overline{AB}$

   $\overline{CD}$ bisects the vertex angle

   Prove: $\overline{CD}$ is a median to the base.

   Write a paragraph, two-column, or flow-chart proof.



This proof leads to the following theorem for isosceles triangles.

Isosceles Triangle Theorem

The bisector of the vertex angle of an isosceles triangle is the same segment as the median to the base.

4. Given: ΔFRD with vertices F(-4,1), R(-1,3), and D(-3,5)
ΔTSU with vertices T(1,1), S(4,3), and U(2,5)
$\angle F \cong \angle T$ and $\angle D \cong \angle U$

a) Graph each triangle in the coordinate plane.
b) Show $\Delta FRD \cong \Delta TSU$.