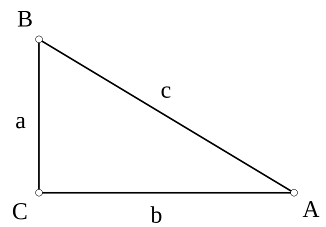
THE PYTHAGOREAN THEOREM

The Pythagorean Theorem is one of the most well-known and widely used theorems in mathematics. We will first look at an informal investigation of the Pythagorean Theorem, and then apply this theorem to find missing sides of right triangles as well as the distance between two points. Finally, we will provide proofs of the Pythagorean Theorem that apply to a variety of learning styles.

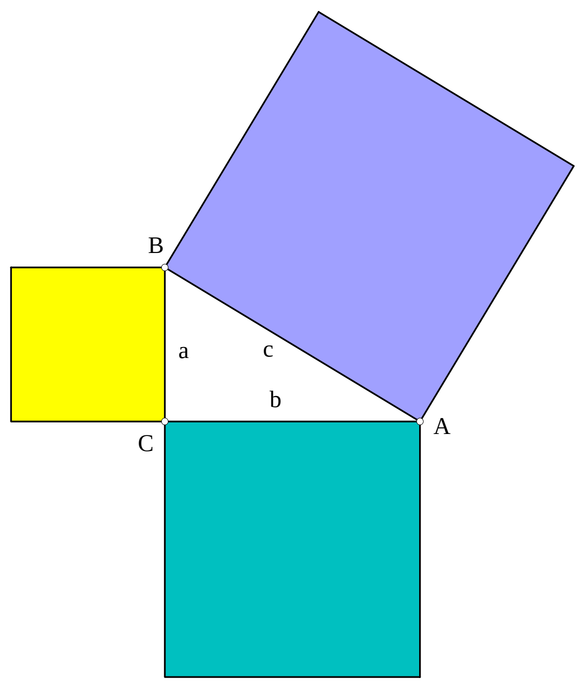
Let us now turn to the classic illustration of the Pythagorean Theorem, and perform some measurements as an investigation. (The diagrams and measurements from this illustration have been created using *The Geometer's Sketchpad* software.)

First, we create right triangle ΔABC. The convention for naming sides is that side *a* is located across from $\angle A$, side *b* is located across from $\angle B$, and side *c* is located across from $\angle C$. We then find the length of its sides, rounded to the nearest hundredth.



a = 2.41 cm
b = 4.00 cm
c = 4.66 cm

Next, we construct squares on the three sides of ΔABC. We then measure the area of each of the squares (to the nearest hundredth), as shown below. What do you notice?



Area of Square at side a = 5.80 cm^2
Area of square at side b = 15.96 cm^2
Area of square at side c = 21.76 cm^2

Notice that the sum of the areas of the two smaller squares equals the area of the largest square; this leads us to the more formal wording of the Pythagorean Theorem:

<u>The Pythagorean Theorem</u>

In a right triangle, the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse.

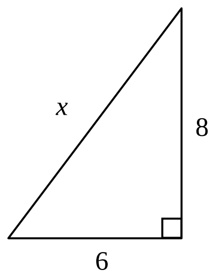
In other words, if *a* and *b* represent the lengths of the legs of a right triangle, and *c* represents the length of the hypotenuse, the Pythagorean Theorem states that:

$$a^2 + b^2 = c^2$$

Examples

Find *x*. Write all answers in simplest radical form.

1.

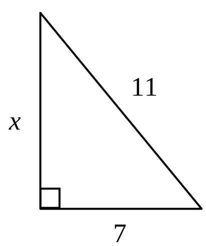


<u>Solution:</u>

The lengths of the legs are 6 and 8, and the length of the hypotenuse is *x*, so

$$6^2 + 8^2 = x^2$$
$$36 + 64 = x^2$$
$$100 = x^2$$
$$10 = x$$
$$x = 10$$

2.

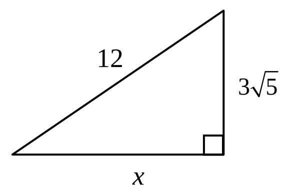


<u>Solution:</u>

The lengths of the legs are 7 and *x*, and the length of the hypotenuse is 11, so

$$7^2 + x^2 = 11^2$$
$$49 + x^2 = 121$$
$$x^2 = 72$$
$$x = \sqrt{72}$$
$$x = \sqrt{36} \cdot \sqrt{2}$$
$$x = 6\sqrt{2}$$

3.



<u>Solution:</u>

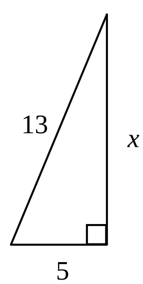
The lengths of the legs are x and $3\sqrt{5}$, and the length of the hypotenuse is 12, so

$$x^2+\left(3\sqrt{5}\right)^2=12^2$$
$$x^2+\left(3\cdot\sqrt{5}\cdot 3\cdot\sqrt{5}\right)=144$$
$$x^2+9\cdot 5=144$$
$$x^2+45=144$$
$$x^2=99$$
$$x=\sqrt{99}$$
$$x=\sqrt{9}\cdot\sqrt{11}$$
$$x=3\sqrt{11}$$

Exercises

Find x. Write all answers in simplest radical form.

1.



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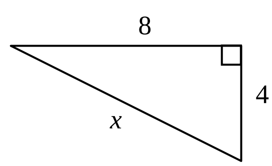

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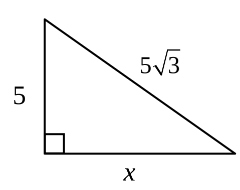


The Pythagorean Theorem and the Distance Formula

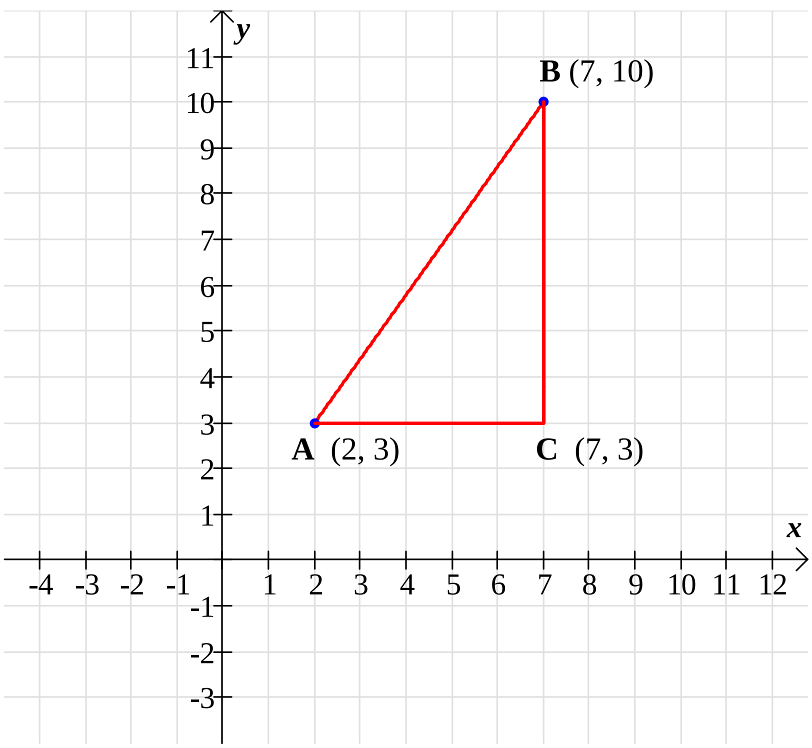
The Pythagorean Theorem can be used to find the distance between two points, as shown below.

Examples

1. Use the Pythagorean Theorem to find the distance between the points A(2, 3) and B(7, 10). Write your answer in simplest radical form.

2. Use the Pythagorean Theorem to find the distance between the points A(-3, 4) and B(5, -6). Write your answer in simplest radical form.

<u>Solution to #1:</u>
We first plot the points A(2, 3) and B(7, 10) on the coordinate plane. We want to find the distance AB. Next, we draw a right triangle ABC that has hypotenuse $\overline{AB}$, as shown below.



We can easily find the lengths of the legs of the triangle:

$AC = 5$ (We can quickly count the units or take the absolute value of the difference of the *x*-coordinates: $|7-2| = 5$, or equivalently $|2-7| = 5$.)

$BC = 7$ (We can quickly count the units or take the absolute value of the difference of the *y*-coordinates: $|10-3| = 7$, or equivalently $|3-10| = 7$.)

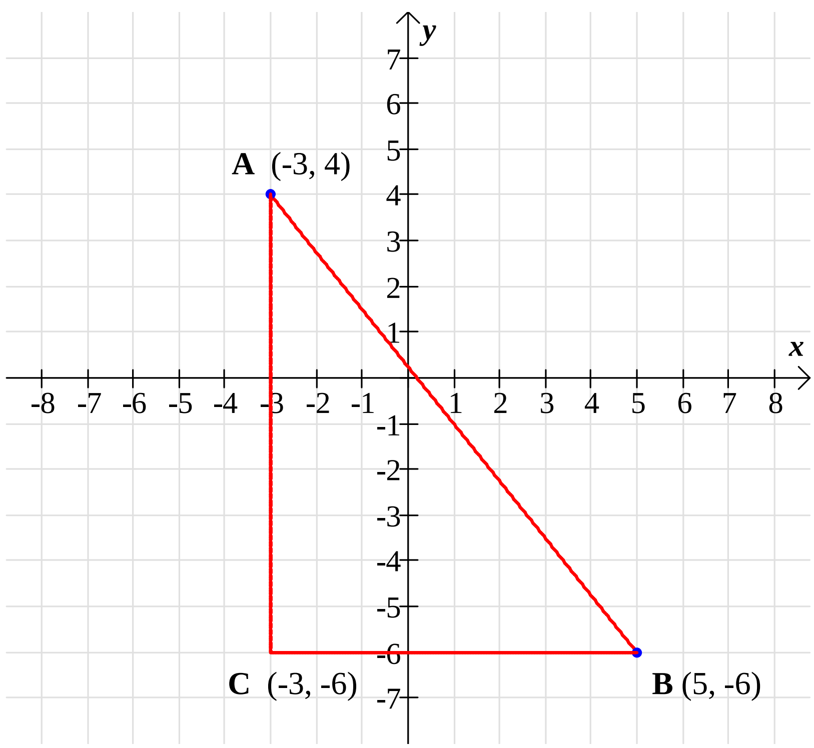
We can now use the Pythagorean Theorem:

$$5^2 + 7^2 = (AB)^2$$
$$25 + 49 = (AB)^2$$
$$74 = (AB)^2$$
$$AB = \sqrt{74}$$

(Note: This square root can not be simplified any further. For additional information on simplifying square roots, refer to the "Irrational Numbers" tutorial in the appendix)

Solution to #2:

We first plot the points A(-3, 4) and B(5, -6) on the coordinate plane. We want to find the distance AB. Next, we draw a right triangle ABC that has hypotenuse $\overline{AB}$, as shown below.



We can easily find the lengths of the legs of the triangle:

$BC = 8$	(We can quickly count the units or find the absolute value of the difference of the *x*-coordinates: $\lvert 5-(-3)\rvert = 8$, or equivalently $\lvert (-3)-5\rvert = 8$.)
$AC = 10$	(We can quickly count the units or find the absolute value of the difference of the *y*-coordinates; $\lvert (-6)-4\rvert = 10$, or equivalently $\lvert 4-(-6)\rvert = 10$.)

We can now use the Pythagorean Theorem:

$$8^2 + 10^2 = (AB)^2$$
$$64 + 100 = (AB)^2$$
$$164 = (AB)^2$$
$$AB = \sqrt{164}$$
$$AB = \sqrt{4} \cdot \sqrt{41}$$
$$AB = 2\sqrt{41}$$

(Note: For additional information on simplifying square roots, refer to the "Irrational Numbers" tutorial in the appendix)

Now that we have used the Pythagorean Theorem, we will solve the same problems using the distance formula instead. The distance formula is shown below. (An informal derivation will be shown later in this section.)

<u>The Distance Formula</u>

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the distance *d* between *A* and *B* is

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

Examples

1. Use the distance formula to find the distance between the points A(2, 3) and B(7, 10). Write your answer in simplest radical form.
2. Use the distance formula to find the distance between the points A(-3, 4) and B(5, -6). Write your answer in simplest radical form.

Solution to #1:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(7-2)^2 + (10-3)^2}$$
$$d = \sqrt{5^2 + 7^2}$$
$$d = \sqrt{25+49}$$
$$d = \sqrt{74}$$

Solution to #2:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(5-(-3))^2 + (-6-4)^2}$$
$$d = \sqrt{8^2 + (-10)^2}$$
$$d = \sqrt{64+100}$$
$$d = \sqrt{164} = \sqrt{4} \cdot \sqrt{41} = 2\sqrt{41}$$

Notice how the numbers in this solutions above correspond to those in the previous set of examples where we used the Pythagorean Theorem directly.

We will now show an informal derivation of the distance formula.

$|x_2 - x_1|$ represents the length of one leg of the triangle.

$|y_2 - y_1|$ represents the length of the other leg of the triangle.

Since *d* represents the hypotenuse of the triangle, we can use the Pythagorean Theorem and obtain the following formula:

$$(|x_2 - x_1|)^2 + (|y_2 - y_1|)^2 = d^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Examples

1. Given the points A(3, 4) and B(7, 12),

   a) Use the Pythagorean Theorem to find the distance from A to B. Write your answer in simplest radical form.

   b) Verify your answer in part (a) by using the distance formula.

2. Given the points A(-1, 7) and B(-4, -2),

   a) Use the Pythagorean Theorem to find the distance from A to B. Write your answer in simplest radical form.

   b) Verify your answer in part (a) by using the distance formula.

Proofs of the Pythagorean Theorem

We will now present four proofs of the Pythagorean Theorem. These particular proofs have been chosen because as a group, they accommodate a variety of learning styles. The learning styles are described below.

> Visual
> Visual learners learn by seeing. They learn best from diagrams, charts, detailed written explanations, and by taking detailed notes themselves.
>
> Auditory
> Auditory learners learn by listening. When dealing with written information, they learn much better when they can hear the information being explained to them.
>
> Tactile/Kinesthetic
> Tactile/Kinesthetic learners learn by doing. They learn best when they can participate in hands-on activities and demonstrations.

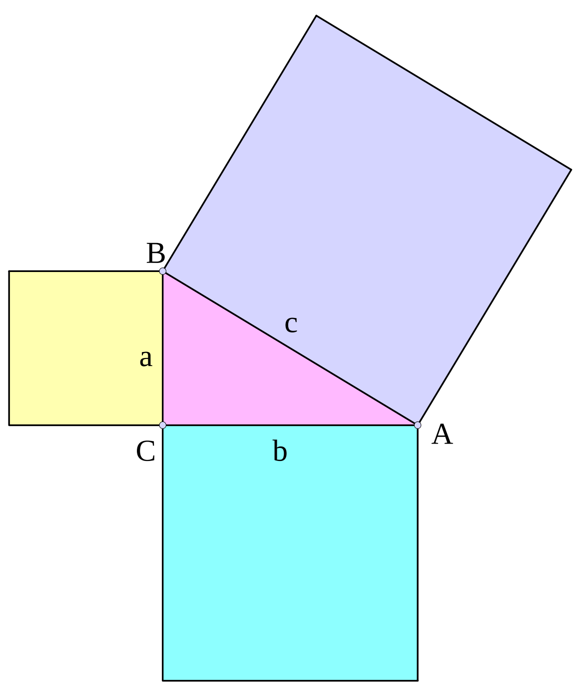
After working through each proof in detail, we will discuss the learning style(s) that apply most to that particular proof.

Proof #1
The following proof is based on the proof by Pythagoras, but we will first set up the proof in detail by asking the reader to help construct some diagrams.

We begin with the classic illustration of the Pythagorean Theorem. We wish to show that

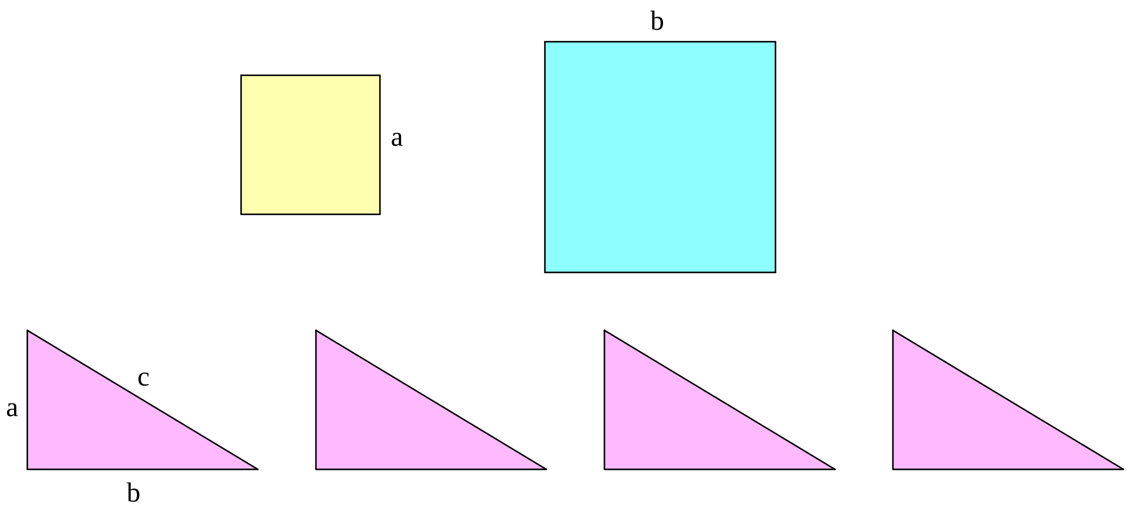
$a^2 + b^2 = c^2$, or in other words,

(the area of the square at side *a*) + (the area of the square at side *b*) = (the area of the square at side *c*)

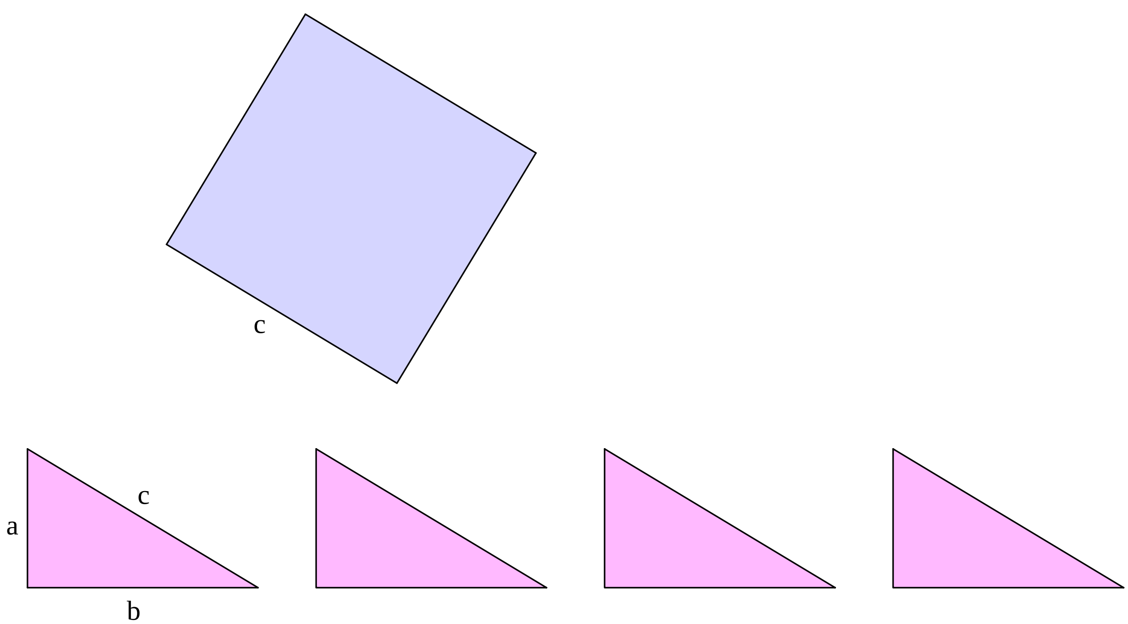


We will now examine the combined area of each of the following groups of polygons.

Group 1: The square at side *a*, the square at side *b*, and four copies of the right triangle (with legs *a* and *b*).

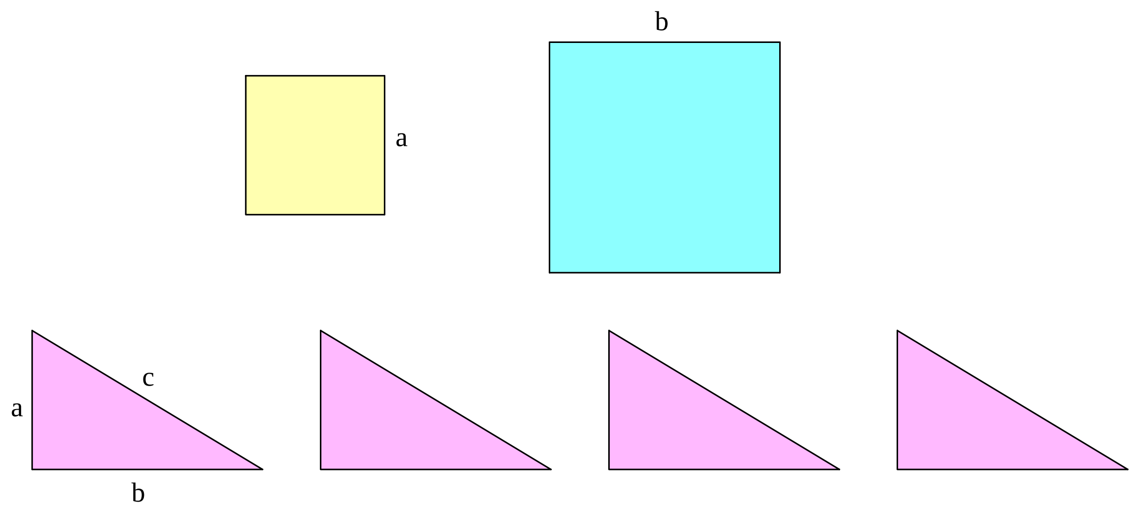


Group 2: The square at side *c* and four copies of the right triangle (with legs *a* and *b*).

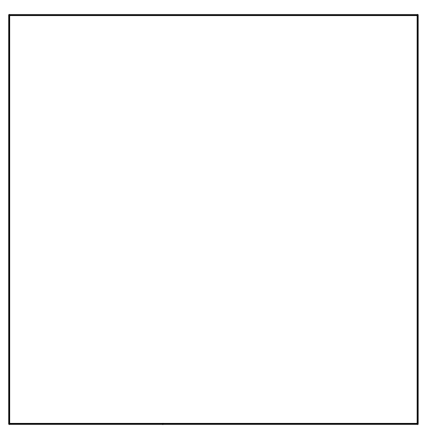


<u>Proof #1: Activity</u>

1. Cut out the six pieces from Group 1 below.



2. Rearrange the six pieces and place them in such a way that they fit exactly into the following square of side length $a+b$.



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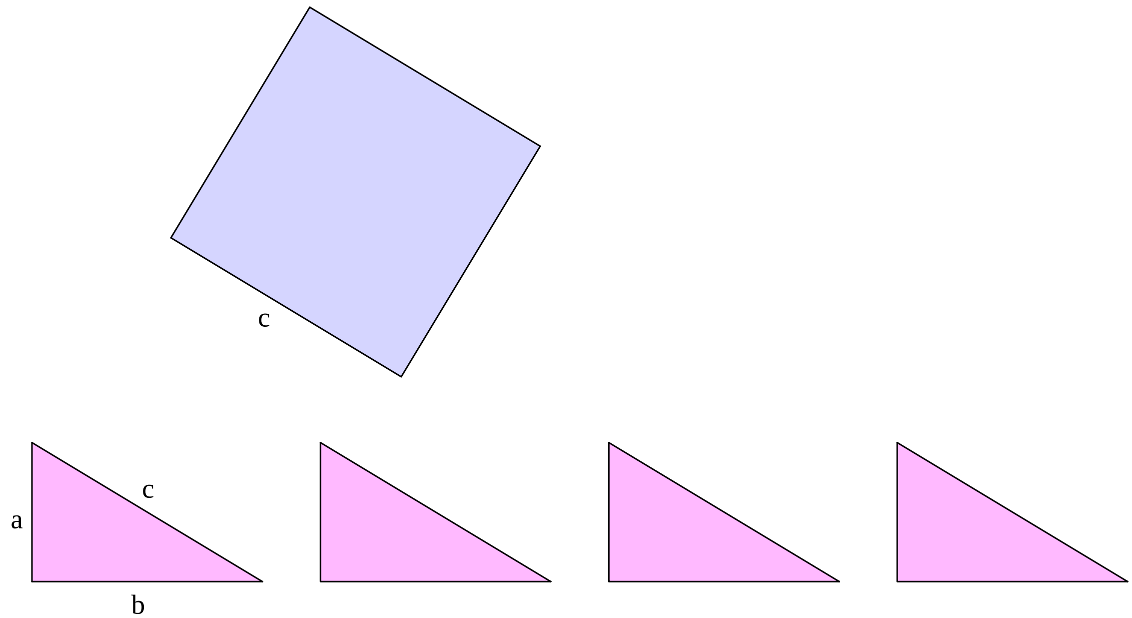
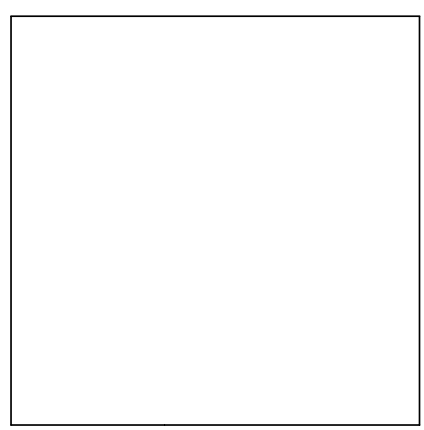
3. Cut out the five pieces from Group 2 below.



4. Rearrange the six pieces and place them in such a way that they fit exactly into the following square of side length $a+b$.



Note: An extra copy of this activity can be found on the next two pages (since this one will be all sliced up after you complete the activity). The solution to the activity can be found on the page after that… No peeking until you have figured it out yourself!!! ☺

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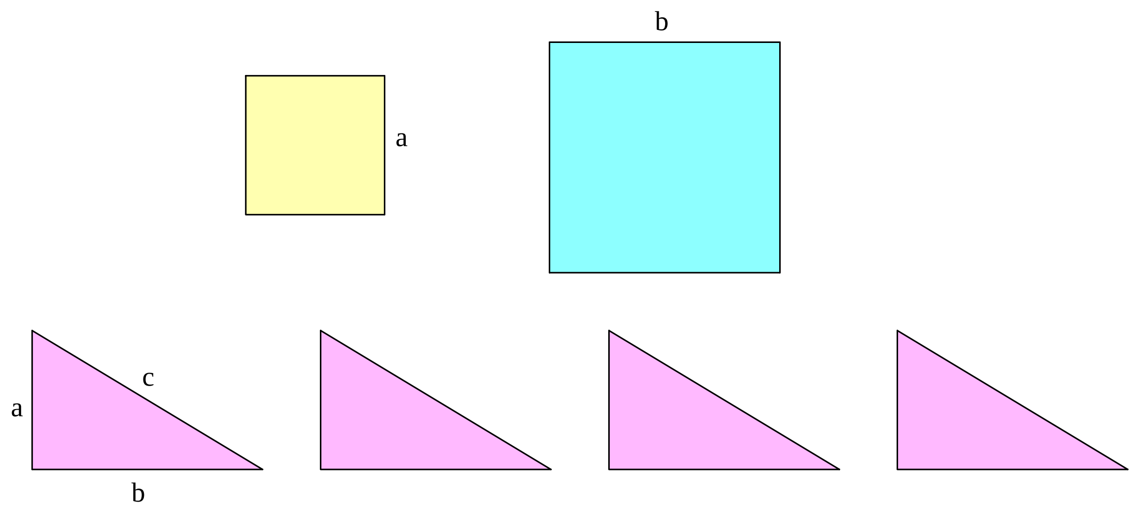
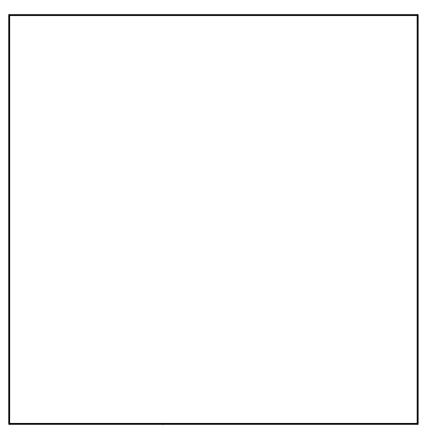
Proof #1: Activity

1. Cut out the six pieces from Group 1 below.



2. Rearrange the six pieces and place them in such a way that they fit exactly into the following square of side length $a + b$.



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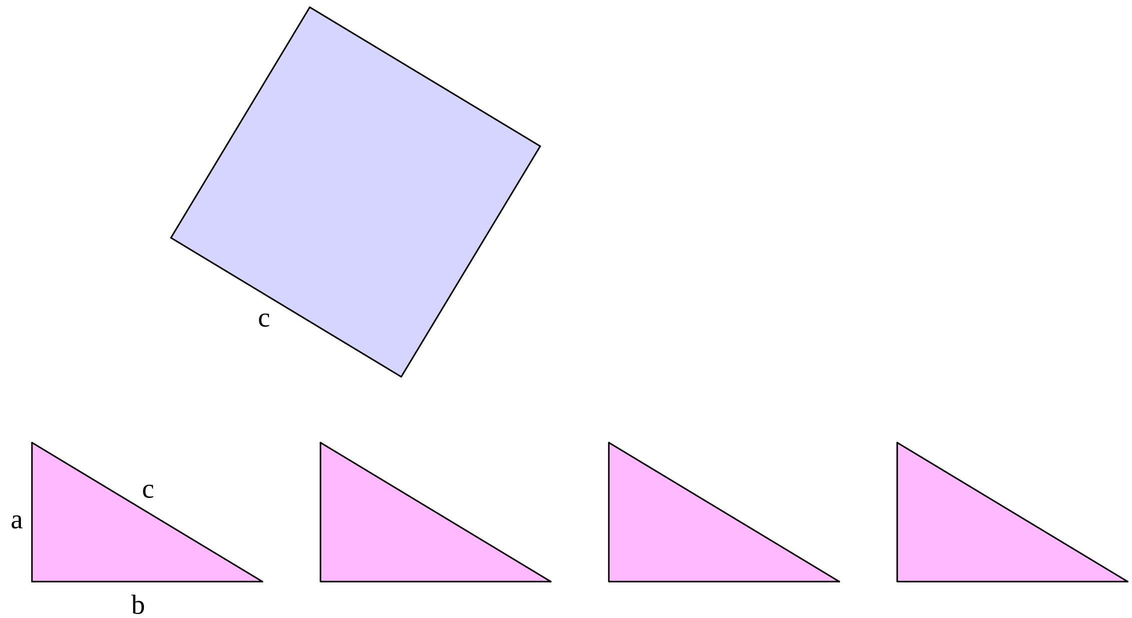
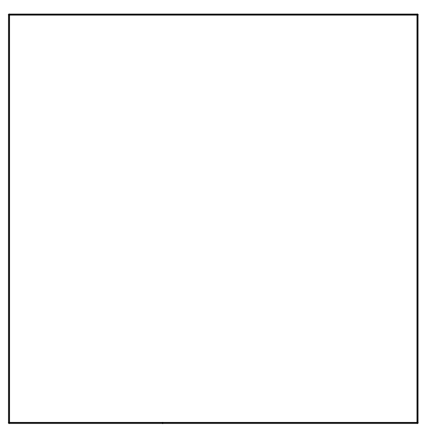


4. Rearrange the five pieces and place them in such a way that they fit exactly into the following square of side length $a+b$.



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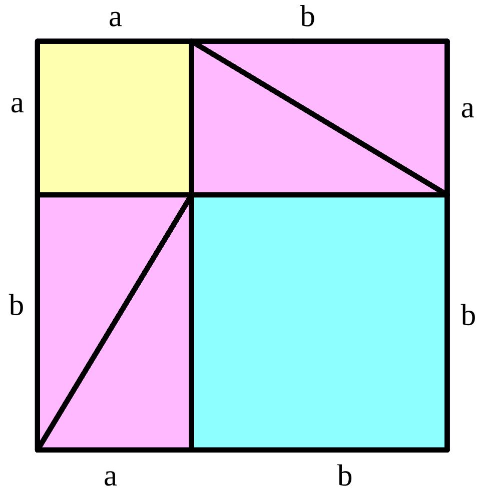
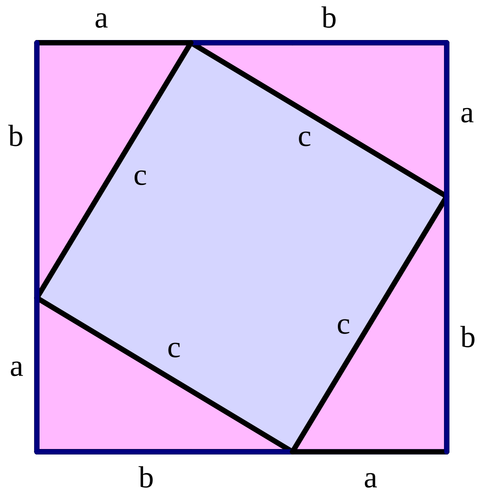
Solution to the Activity for Proof #1:

Arrangement of Group 1 Objects:



There are slight variations of the arrangement of the objects from Group 1, but all solutions have the following characteristics: The square with side *a* must be placed in one corner, the square with side *b* must be placed in the opposite corner, and the four triangles must be placed in pairs to form two rectangles to comprise the remainder of the diagram.

Arrangement of Group 2 Objects:



We will learn on the next page why this result leads us to the conclusion of the Pythagorean Theorem…

Justification for Proof #1:

We first find the area of each of the individual objects in Group 1:

Area of square with side *a:* a^2
Area of square with side *b*: b^2
Area of one triangle: $\frac{1}{2}ab$ (There are four such triangles in the diagram.)

We can now find the combined area of the Group 1 objects by adding the individual areas together:

Combined area of Group 1 Objects = $a^2 + b^2 + 4\left(\frac{1}{2}ab\right) = a^2 + b^2 + 2ab$

We then find the area of each of the individual objects in Group 2:

Area of square with side *c:* c^2
Area of one triangle: $\frac{1}{2}ab$ (There are four such triangles in the diagram.)

We can now find the combined area of the Group 2 objects by adding the individual areas together:

Combined area of Group 2 Objects = $c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$

The combined area of the Group 1 objects is equal to the combined area of the Group 2 objects (since they both comprise a square of length $a+b$). Therefore, we can set the following quantities equal:

Combined area of Group 1 Objects = Combined area of Group 2 Objects

$$a^2 + b^2 + 2ab = c^2 + 2ab$$

We can then subtract the quantity $2ab$ from both sides of the equation (which is equivalent to removing the four triangles from each of our diagrams). We obtain the following result, which is the Pythagorean Theorem:

$$\boxed{a^2 + b^2 = c^2}$$

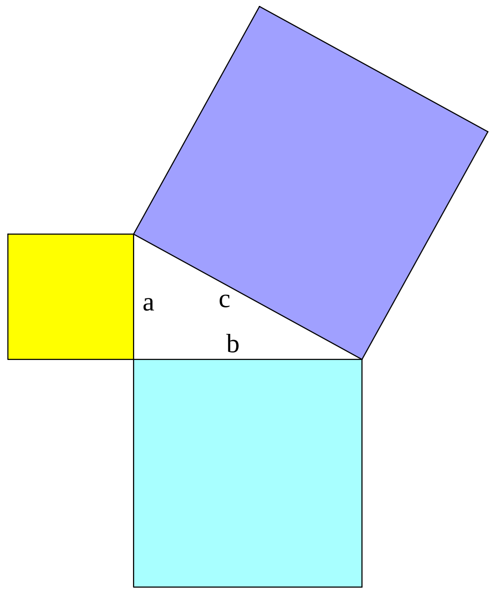
Learning Style Analysis: Proof #1 would appeal primarily to the Kinesthetic learner. If presented in a lecture format, the explanations would appeal to the Auditory learner. The diagrams and equations in this proof would also appeal to the Visual learner.

Proof #2

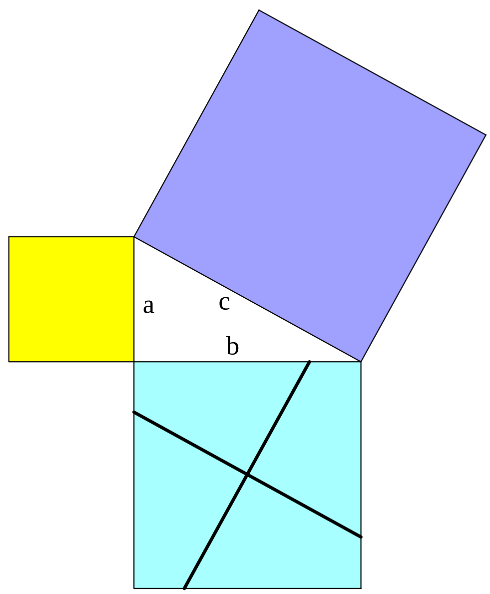
We begin with the classic illustration of the Pythagorean Theorem. We wish to show that

$$a^2 + b^2 = c^2 \text{, or in other words,}$$

(the area of the square at side *a*) + (the area of the square at side *b*) = (the area of the square at side *c*)



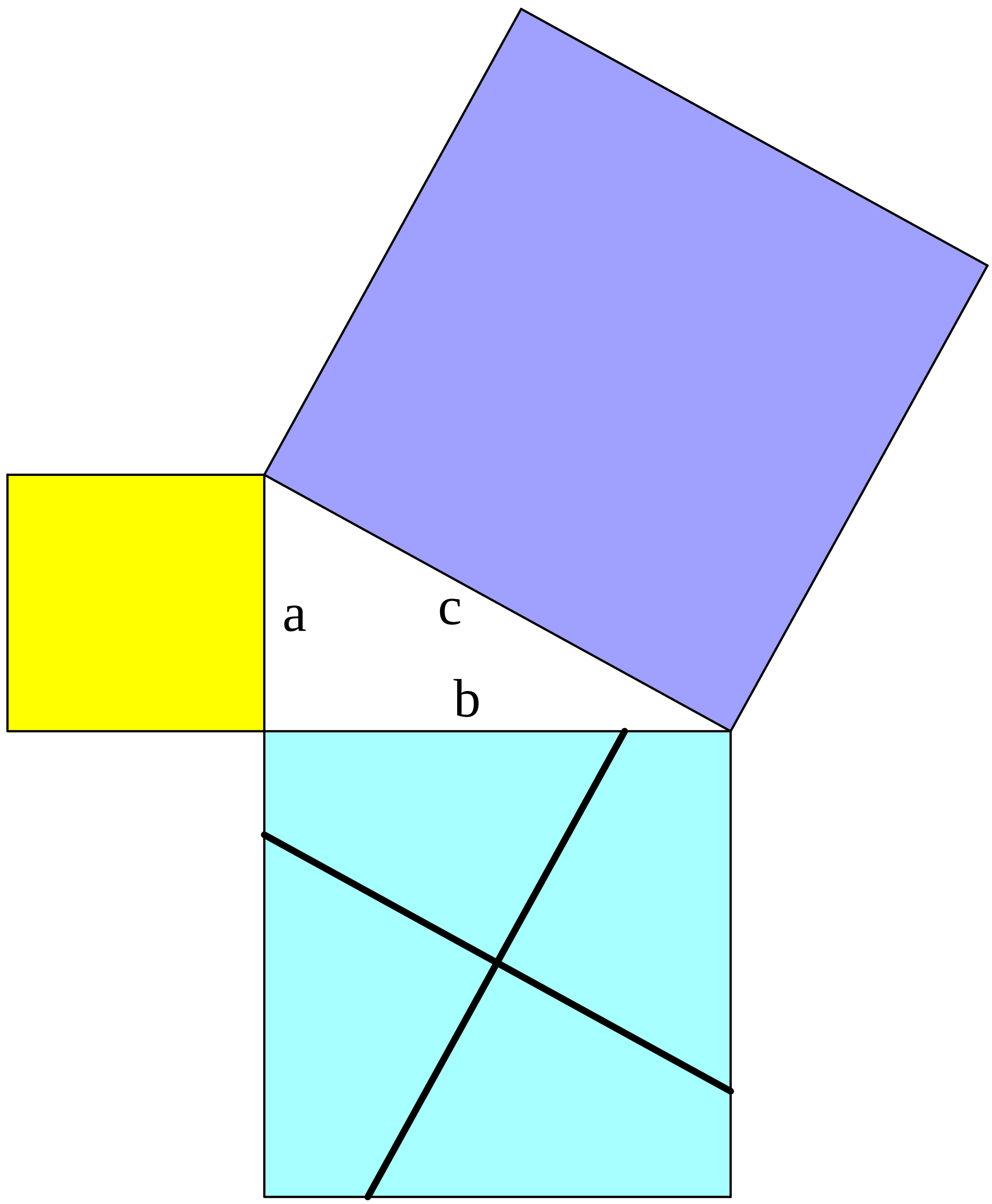
We will now construct two lines through the center of the square at side *b*; one which is parallel to side *c*, and the other which is perpendicular to side *c*, as shown below.



Proof #2: Activity

1. Cut out the square at side *a*.
2. Cut out the square at side *b*, and then cut along the thick lines inside this square to divide it into four pieces
3. Rearrange the five pieces (from squares a^2 and b^2) and place them in such a way that they fit exactly into the square at side *c*.

Note: An extra copy of this activity can be found on the next page (since this one will be all sliced up after you complete the activity). The solution to the activity can be found on the page after that… No peeking until you have figured it out yourself!!! ☺



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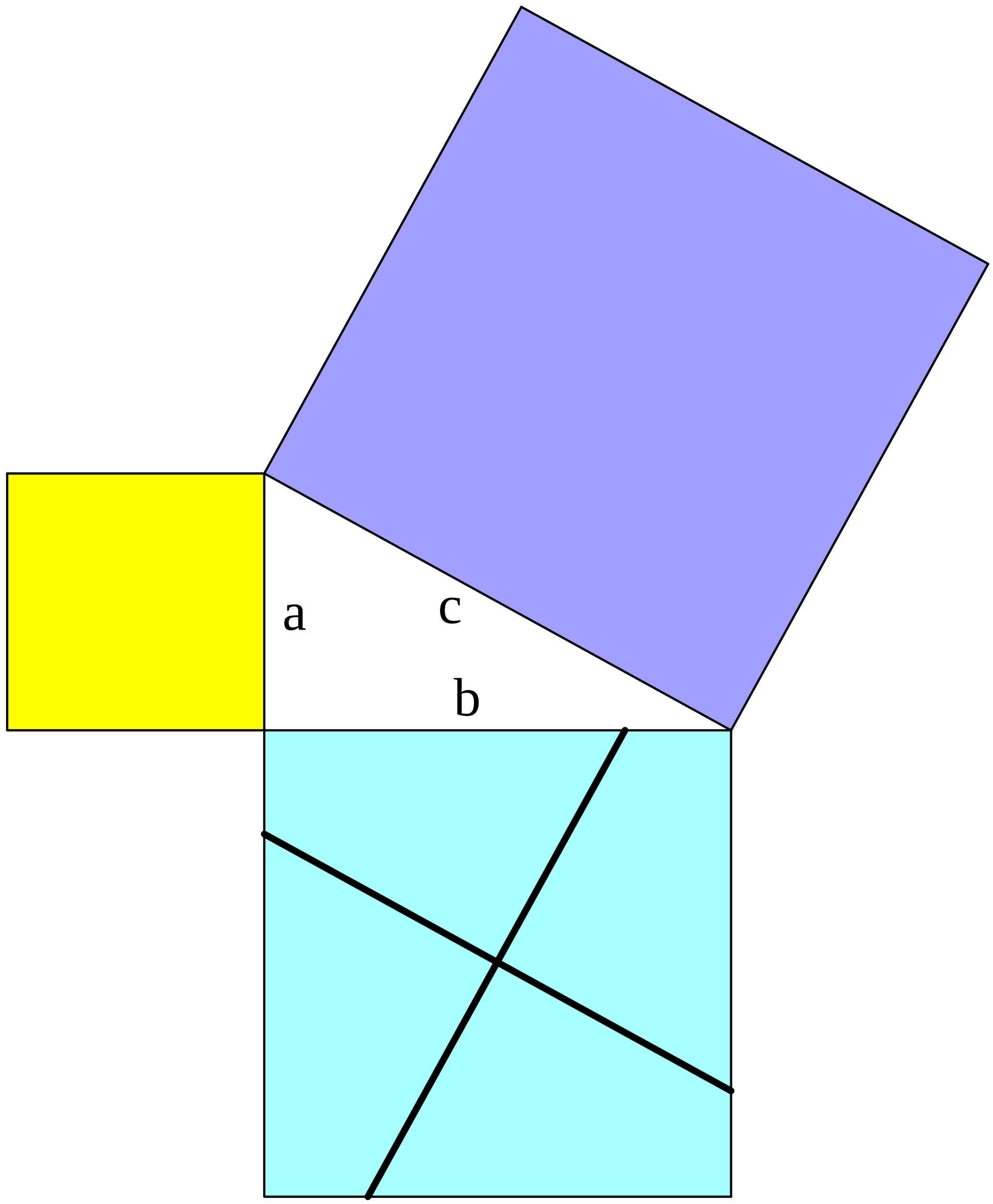
Now perform the following activity:

1. Cut out the square at side *a*.
2. Cut out the square at side *b*, and then cut along the thick lines inside this square to divide it into four pieces
3. Rearrange the five pieces (from squares a^2 and b^2) and place them in such a way that they fit exactly into the square at side *c*.



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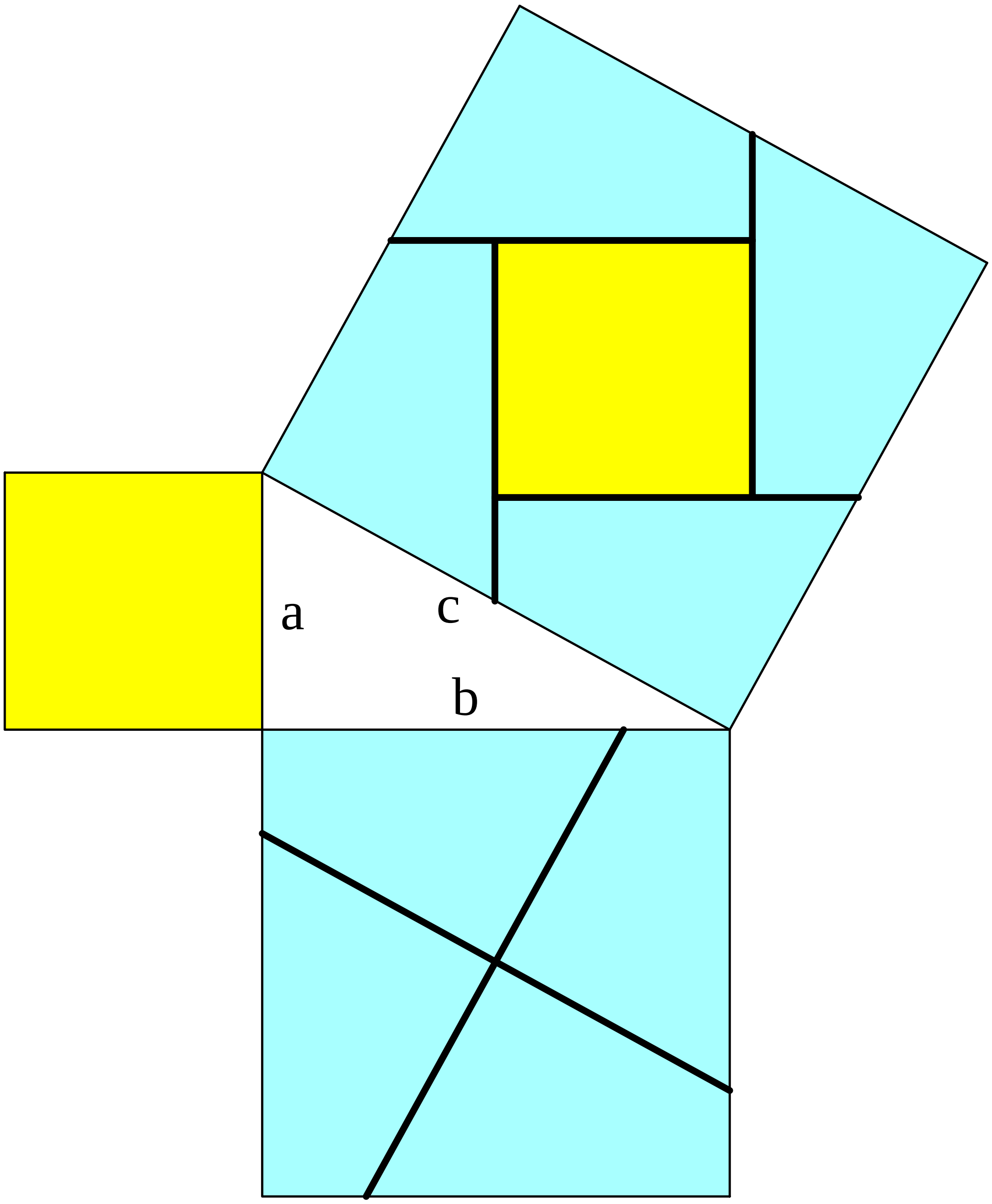
Solution to Proof #2 (Perigal's Proof)

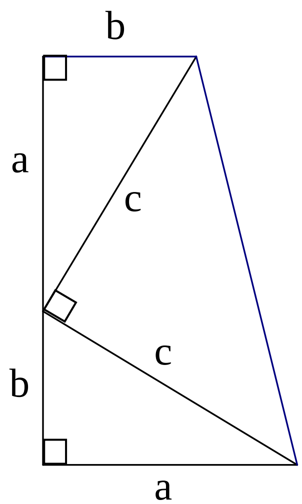


The previous proof is known as a dissection proof, and is attributed to Henry Perigal, a London stockbroker who published the proof in 1873 (Bogomolny). The algebraic justification of this proof is lengthy and will not be shown in this text.

Learning Style Analysis: Proof #2 would appeal primarily to the Kinesthetic learner. As with all of our proofs of the Pythagorean Theorem, the diagrams would also appeal to the Visual learner.

Proof #3
The following proof was discovered by President Garfield in 1876 (Bogomolny).

Garfield's proof is based on the formula for the area of a trapezoid. His diagram is shown below.



If we find the area of the figure by using the formula for the area of a trapezoid,

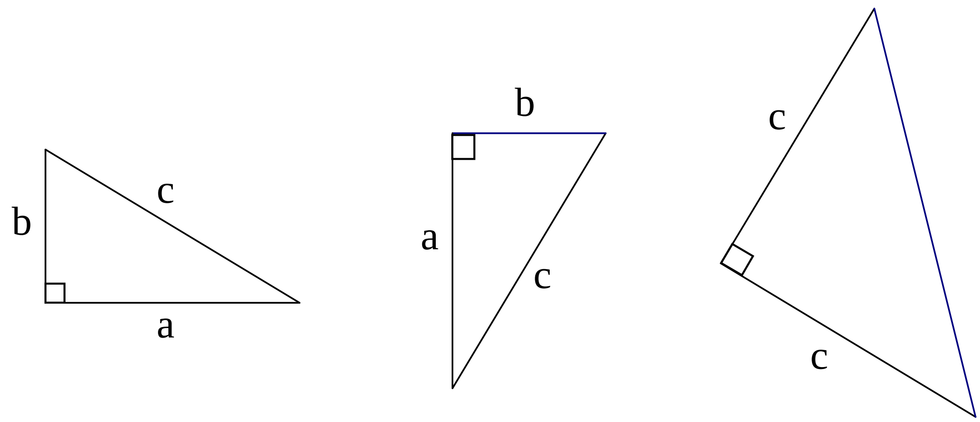
$$Area\ of\ trapezoid = \frac{h}{2}\left(base_1 + base_2\right)$$

$$Area\ of\ trapezoid = \frac{a+b}{2}\left(a+b\right)$$

$$Area\ of\ trapezoid = \frac{a^2+2ab+b^2}{2} = \frac{a^2}{2}+\frac{2ab}{2}+\frac{b^2}{2} = \tfrac{1}{2}a^2+ab+\tfrac{1}{2}b^2$$

Now we find the area of each of the individual triangles, and then add them up to find the area of the entire figure:

Remember that the formula for the area of a triangle is $A = \frac{1}{2}bh$.



<u>Triangle #1</u>

$A_1 = \frac{1}{2}ab$

<u>Triangle #2</u>

$A_2 = \frac{1}{2}ab$

<u>Triangle #3</u>

$A_3 = \frac{1}{2}c^2$

Adding up the areas of the three triangles, we obtain

$$\text{Total area of triangles} = A_1 + A_2 + A_3 = \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}c^2 = ab + \tfrac{1}{2}c^2$$

We now set the area of the trapezoid equal to the total area of the triangles:

$$\text{Area of trapezoid} = \text{Total area of triangles}$$

$$\tfrac{1}{2}a^2 + ab + \tfrac{1}{2}b^2 = ab + \tfrac{1}{2}c^2$$

We then subtract *ab* from both sides, and obtain the equation

$$\tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 = \tfrac{1}{2}c^2$$

Finally, we multiply both sides by 2, and obtain our desired result:

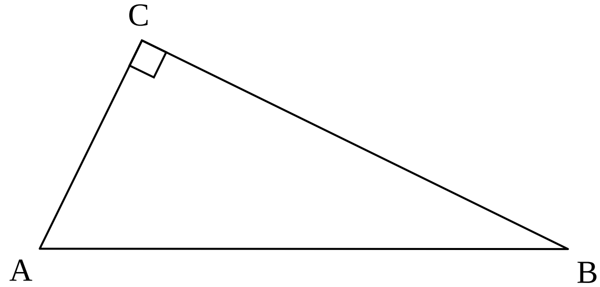
$$a^2 + b^2 = c^2.$$

<u>Learning Style Analysis:</u> The diagrams and equations in Proof #3 would appeal to the Visual learner. If Proof #3 were to be presented in a lecture format, the explanations would appeal to the Auditory learner.

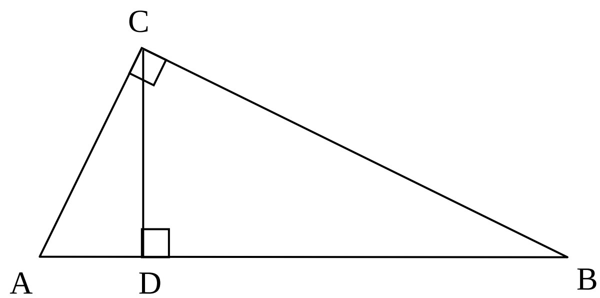
Preface to Proof #4

We need to preface our first proof with a discussion about similarity.

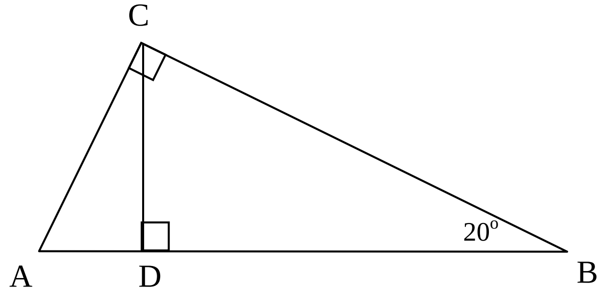
First, we draw right triangle ABC.



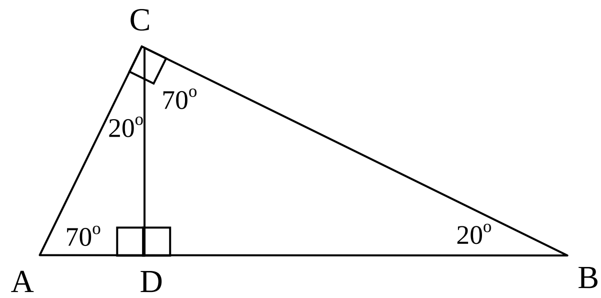
Next, we draw an altitude from vertex C to the hypotenuse, $\overline{AB}$, and we label the point of intersection *D*. Notice that the diagram below contains three triangles; the original triangle as well as two smaller triangles that have been formed by the altitude.



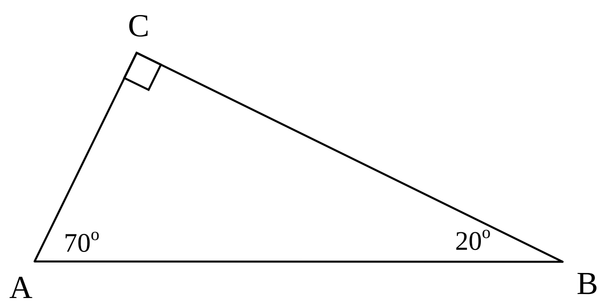
Let us suppose, for the purpose of illustration, that $\angle B$ measures 20°. We then want to find the measures of all the other angles in the diagram. (Note that the diagram may not be drawn to scale.)

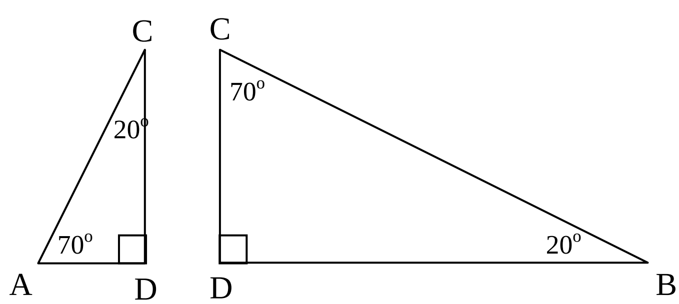


Since the sum of the measures of the angles of a triangle is 180°, we can quickly obtain the following angle measures.



We now draw the three triangles separately and label the measures of their angles.





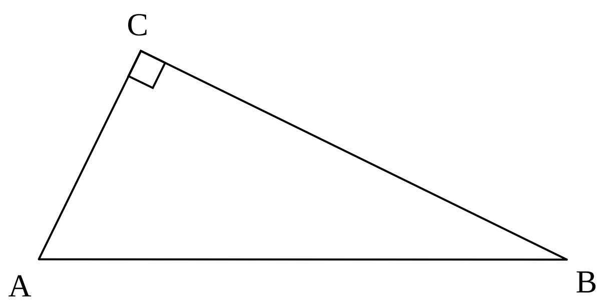
If three angles of one triangle are congruent to three corresponding angles of another triangle, then the triangles are similar. (This is called AA Similarity, since it is actually sufficient to just show that two angles of one triangle are congruent to two corresponding angles of another triangle -- since the third angles must then be congruent as well.) Since each of the above triangles have the same set of angle measurements, the three triangles are similar to each other. We must carefully match up the corresponding angles as we name them in the following similarity statement. (We have chosen to name the 70° angle first, then the 20° angle, then the 90° angle.):

$$\Delta ABC \sim \Delta ACD \sim \Delta CBD \qquad \text{(The symbol } \sim \text{ means similar.)}$$

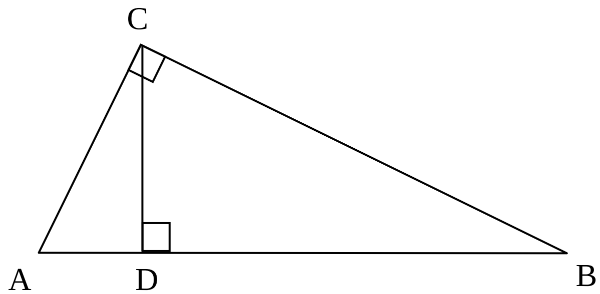
The 20^o and 70^o angles were used solely for the purpose of illustration; the above example can be generalized to any right triangle, as will be shown in Proof #4. The similar triangles will be used to prove the Pythagorean Theorem.

Proof #4

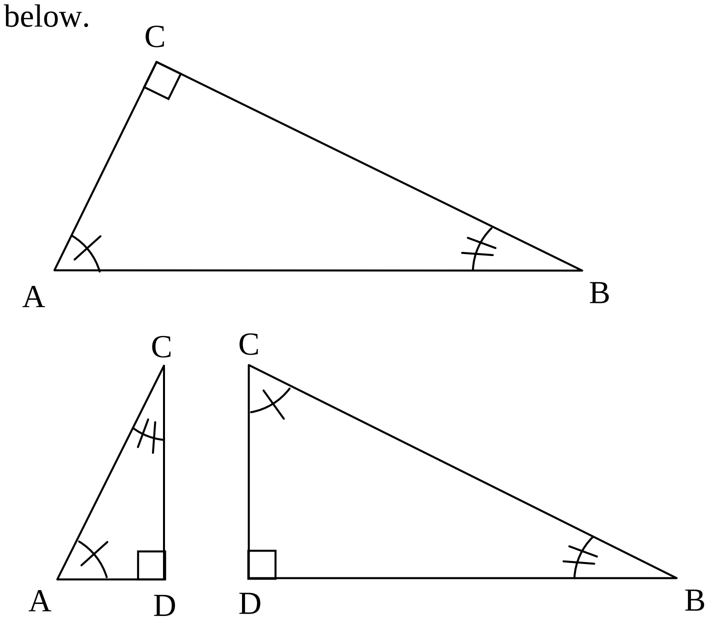
We begin with right triangle *ABC*. We need to prove that $(AC)^2 + (BC)^2 = (AB)^2$.



We then draw an altitude to the hypotenuse, with point of intersection *D*.



There are three different triangles in the diagram above. Their congruent angles are marked below.



The three triangles are similar by AA Similarity; therefore

$$\Delta ABC \sim \Delta ACD \sim \Delta CBD$$

Since the triangles are similar, their corresponding sides are proportional.

Using the fact that $\Delta ABC \sim \Delta ACD$, we can say that $\dfrac{AC}{AB} = \dfrac{AD}{AC}$.

Cross multiplying, we obtain the equation $AC \cdot AC = AB \cdot AD$. (1)

Using the fact that $\Delta ABC \sim \Delta CBD$, we can say that $\dfrac{BC}{AB} = \dfrac{BD}{BC}$.

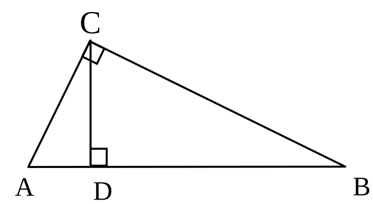
Cross multiplying, we obtain the equation $BC \cdot BC = AB \cdot BD$. (2)

Adding equations (1) and (2), we obtain

$$AC \cdot AC + BC \cdot BC = AB \cdot AD + AB \cdot BD$$

$$(AC)^2 + (BC)^2 = AB(AD + BD)$$

Since $AD + BD = AB$ (see diagram at right),



$(AC)^2 + (BC)^2 = AB \cdot AB$, so

$$(AC)^2 + (BC)^2 = (AB)^2$$

<u>Learning Style Analysis:</u> If Proof #4 were to be presented in a lecture format, the explanations would appeal to the Auditory learner. The diagrams and equations would appeal to the Visual learner.

Online Resource

The following webpage contains an award-winning Java applet which was written by Jim Morey. This applet is an excellent visual demonstration of the Pythagorean Theorem: http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Pythagoras/pythagoras.html

Works Cited

Bogomolny, Alexander. *Pythagorean Theorem*. 2004. CTK Software, Inc. <webster.commnet.edu/mla/online.shtml>