Inductive Versus Deductive Reasoning

Inductive reasoning is a method of drawing conclusions based upon limited information. In essence, the phrase “inductive reasoning” is a sophisticated substitute for the word “guessing”. For example, if we know the first five terms of a sequence are given by

$$2, 4, 6, 8, 10$$

then we might *guess* that the next term in the sequence is 12. We do not know for sure that the next term is 12. We make this conclusion based upon our knowledge of the first five terms. We will not know if our conclusion is correct until we see the sixth term in the sequence. If the sixth term is 25, then our conclusion is false. If the sixth term is 12, then the conclusion is true. Regardless, our conclusion is nothing more than a guess until we actually see the sixth term.

Example: A table of numbers is shown below.

n	2^n+1
1	3
2	5
4	17
8	257
16	65,537

The numbers in the first column are powers of 2 since $2^0=1$, $2^1=2$, $2^2=4$, $2^3=8$ and $2^4=16$. The numbers in the second column are the result of applying the formula 2^n+1 to the numbers in the first column. A famous mathematician named Pierre de Fermat computed the values in this table in the 1600s and he noticed that the numbers 3, 5, 17, 257 and 65,537 are prime numbers. No doubt he had to do many calculations to verify that the final number was a prime number since no computers were available at that time. Based upon his calculations, Fermat used **inductive reasoning** to conjecture that every number of the form 2^n+1 is a prime number whenever *n* is a power of 2. The next power of 2 in the first column of the table about should be 32, so the next number in the second column above should be $2^{32}+1=4{,}294{,}967{,}297$. It wasn’t until 100 years later than a mathematician named Leonhard Euler proved that the number 4,294,967,297 is divisible by 641, and consequently it is not prime. As a result, the conjecture that resulted from Fermat’s use of **inductive reasoning** is false.

Deductive reasoning is a method of drawing conclusions based upon logic and fact. For example, we can see from the first five terms in the sequence above that the second term is 4. There is no question as to the validity of this conclusion. It is completely based upon fact.

Although **inductive reasoning** can sometimes lead to false conclusions, it can often be a useful first step in the process of applying **deductive reasoning** to determine whether a conclusion is true. We illustrate this with a simple example below.

Example: Pretend for the moment that you are a grade school child. You know the difference between even and odd numbers, and you know how to multiply numbers. One of the activities that your teacher has given you involves multiplying natural numbers, and you notice that every time you multiply an even natural number with another natural number you get an even natural number. Based upon your observations, you use **inductive reasoning** to conclude that the product of an even natural number with a natural number is always an even natural number. Your conclusion is a *guess*, but it might lead you (or someone else) to use **deductive reasoning** to prove the assertion. We can do this by giving a precise statement of the assertion and then supplying a proof based upon logic and facts. To this end, we let $\mathbb{N}$ be the set of natural numbers, and we let E be the set of even natural numbers. That is,

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad E = \{2n \mid n \in \mathbb{N}\}$$

Theorem: If $k, m \in \mathbb{N}$ and k is an even number, then the product km is an even natural number.

Proof: Since k is an even natural number, there is a natural number n so that $k = 2n$. As a result, direct substitution and the associative law for multiplication give

$$km = (2n)m = 2(nm)$$

Since the natural numbers are closed under multiplication, the product nm is a natural number. So, $km = 2(nm)$ is an even natural number. •

Our use of **deductive reasoning** removes any doubt as to the validity of the assertion. There is no need to create a larger multiplication table to check additional values. The assertion is simply true! However, we might not have ever tried to use **deductive reasoning** to prove this assertion without the set of observations that lead to our **inductive reasoning** to guess the result. In general, although **inductive reasoning** can sometimes lead to false conclusions, it is a good tool for making conjectures which may be verified using **deductive reasoning**.