Euclidean Geometry

Students are often so challenged by the details of Euclidean geometry that they miss the rich structure of the subject. We give an overview of a piece of this structure below.

We start with the idea of an **axiomatic system**. An axiomatic system has four parts:

undefined terms
axioms (also called postulates)
definitions
theorems

We will use a slight modification of a subset of axioms developed by *The School Mathematics Study Group* during the 1960's for our structure. This list of axioms is not as brief as one that would be used by graduate students in mathematics, nor is it as extensive as some of the systems used in middle school textbooks.

After we develop the axioms and definitions, we discuss their consequences (theorems). We assume the reader has a working knowledge of Euclidean geometry; we are not building geometry from the axioms. We are deepening and widening an already established body of knowledge.

The Axiomatic Structure for Euclidean Geometry (Points, Lines and Planes)

Most people visualize a point as a tiny dot. Lines are thought of as long, seamless concatenations of points, and planes are thought of as finely interwoven lines: smooth, endless and flat. The actual truth is that these objects may be visualized but they cannot be defined. Any attempt at a definition ends up circling around the terms and using one to define the other. Think of these as the basic syllables in a language – the sounds that make up our language have no meaning in themselves but are combined to make words.

The conventions of the Cartesian plane are well suited to assisting in visualizing Euclidean geometry. However there are some differences between a geometric approach to points on a line and an algebraic one, as we will see in the explanation of Postulate 3.

Axioms

We give an introduction to a subset of the axioms associated with two dimensional Euclidean geometry. Axioms 1 through 8 deal with points, lines, planes, and distance. Axioms 9 through 13 deal with angle measurement and construction, along with some fundamental facts about linear pairs. The axioms related to angle measurement give us a basis for discussing parallel and perpendicular lines. Axiom 14 allows us to complete the discussion of parallel lines and ultimately prove that the sum of the measure of the angles of a triangle is 180 degrees.

The first 8 axioms give us the relationship between points and lines, points and planes, lines and planes, and distance and location. These final notions set the stage for the Cartesian coordinate plane as our model.

A1. Given any two distinct points, there is exactly one line that contains them.

A2. **(The Distance Postulate)** To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

A3. **(The Ruler Postulate)**The points of a line can be placed in a correspondence with the real numbers such that

- A. To every point of the line there corresponds exactly one real number.
- B. To every real number there corresponds exactly one point of the line.
- C. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

A4. **(The Ruler Placement Postulate)** Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

A5.
- A. Every plane contains at least three non-collinear points.
- B. Space contains at least four non-coplanar points.

A6. If two points lie in a plane, then the line containing these points lies in the same plane.

A7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

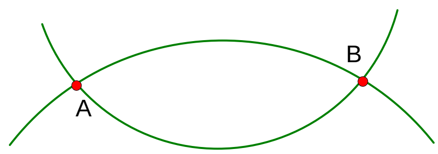
A8. If two planes intersect, then their intersection is a line.

We discuss each of the axioms below.

A1. Given any two distinct points, there is exactly one line that contains them.

Axiom 1 gives a relationship between the undefined terms, point and line.

Exercise: You already have some picture of what is meant by a point and line. Explain why axiom 1 tells us it is impossible to have the following illustration and call the objects points and lines.



A2. (The Distance Postulate) To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

Notice that this axiom does not specify a formula for calculating the distance between two points, it just asserts that this can be done. Nor does it specify any units that are to be used. This allows maximum flexibility in interpreting the axiom, which is a virtue in axiomatic systems. To explore a geometry that uses a non-traditional distance formula, look up Taxicab Geometry on the internet. Taxicab Geometry uses the same axioms as Euclidean Geometry up to Axiom 15 and a very different distance formula.

We need some notation to help us talk about the distance between two points. Whenever A and B are points, we will write AB for the distance from A to B.

Axiom 2 stipulates that the distance between two distinct points is positive – there is no such thing as negative distance. Further, it states that the number is unique. This means that if AB = 5, then there is no other number for the distance. It is impossible for AB = 5 and AB = 3. Notice that the axiom does not say that B is the only point that is 5 away from A; it simply requires that once you establish the distance between these two points, it is fixed.

A3. (The Ruler Postulate) The points of a line can be placed in a correspondence with the real numbers such that

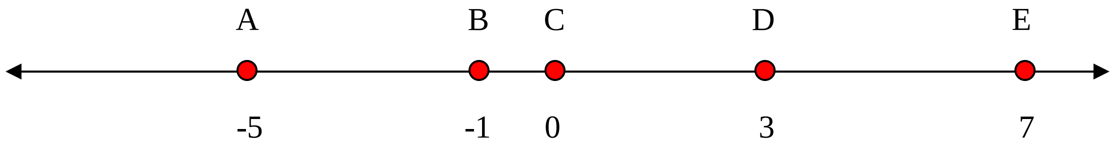
A. To every point of the line there corresponds exactly one real number.

B. To every real number there corresponds exactly one point of the line.

C. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

Most people think of the standard horizontal number line when they read Axiom 3. We will begin our discussion there, but when we begin using the Cartesian coordinate plane, we will see that there is much more to this axiom than an initial reading shows.

Consider the number line shown below. We have identified points A, B, C, D and E on the line with the real numbers A = -5, B = -1, C = 0, D = 3, and E = 7.



The numbers associated with the points above are commonly referred to as **coordinates**.

The distance from E to B is given by the absolute value of the difference of the coordinates in either order.

$$EB = |7-(-1)| = |7+1| = 8$$

Note that EB = BE since we are using the absolute value of the difference. This result is true in general.

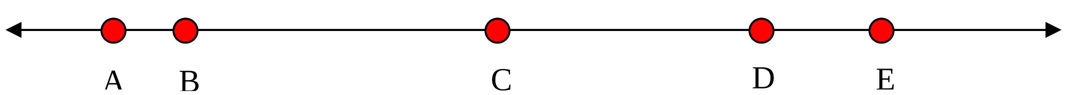
Theorem: If P and Q are points, then PQ = QP.

Exercise: Find AB, AC, AD, AE, BC, BD, CD and DE.

Using absolute value is essential in axiom 3 because the preceding axiom requires the distance to be a positive number; if we want to find distances by subtracting, then we are required to ensure that the subtraction results in a positive number, hence the absolute value in the statement.

A4 (The Ruler Placement Postulate) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Consider the number line below with the five points A, B, C, D and E.

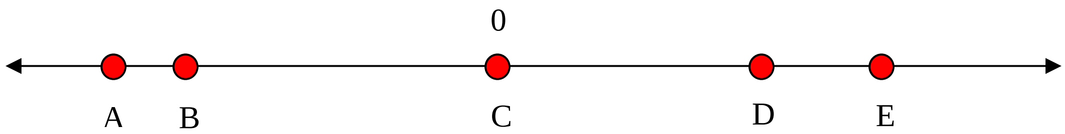


We have already looked at axioms that say there is a distance between each of these points and that we can assign real numbers to each point that work in such a way that the absolute value of the difference of the numbers is the distance. Suppose we have the following distances between the points:

AB = ½, BC = 3, CD = 2.5, DE = 1

Based upon the picture above, what value should be assigned to AE? BD?

Axiom 4 allows us to assign the coordinate 0 to one of these points. Suppose we pick C to have the coordinate zero. We can use the distances above to determine the possible coordinates for the points B, C, D and E.



If we treat the number line above like a standard number line, then the points with negative coordinates should be to the left of C. BC = 3 so B has coordinate –3. AB = ½, so A has coordinate –3.5. CD = 2.5 so the coordinate of D is 2.5. E is located 1 away from D, so E has coordinate 3.5.

Exercise: How will these coordinates change if we chose the point B to have the coordinate 0?

Exercise: Choose the coordinate of C to be 0. Suppose numbers with positive coordinates lie to the left of C. What are the coordinates of the points A, B, D and E?

A5. A. Every plane contains at least three non-collinear points.
B. Space contains at least four non-coplanar points.

Axiom 5 introduces the third undefined term (plane), along with its relationship to points. The term "non-collinear" means "not lying on the same line." Since there are at least 3 non-collinear points to a plane, a plane is much different from a line (see A1).

Exercise: Suppose you take 3 non-collinear points and call them A, B, and C. Assume these points lie to be on the plane of your paper. If you apply A1 to these 3 points, what happens geometric shape is constructed?

Exercise: Take the shape from the exercise above and imagine a point D which is 2 inches above the plane of your paper. Now apply A1 using this point with each of the points A, B and C. The points A, B, C and D determine a solid. Try to give a perspective drawing of this solid.

Remark: A point has 0 dimension, a line has 1 dimension, a plane as a 2 dimensions, and space has 3 dimensions. Notice how your work in the previous exercises started with zero dimensions (creating each of the points A, B and C), progressed to 1 dimension with the creation of each of the line segments connecting these points, progressed to 2 dimensions with the creation of the triangle ABC, and finally progressed to 3 dimensions with the creation of the solid determined by ABCD.

A6. If two points line in a plane, then the line containing these points lies in the same plane.

Axiom 6 gives the relationship between planes and lines. It ties A1 and A5 together.

A7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

If you take any 3 points, it is possible that they are collinear.

Exercise: How many planes pass through 3 collinear points? We can illustrate your answer with a piece of paper. Fold the paper in half along its length and put a line with 3 points showing along the fold line. Leave the paper folded. Do you see that the plane of the paper is one plane containing the points? Now hold the paper up and look at it with the line pointing in between your eyes, support the line with your fingers and let the sides of the paper drop naturally. Do you see that the halves of the paper can be viewed as the visible parts of two planes, one on the left going straight through the line and one on the right going straight through the line? Now shift the edges of the paper – you have two more planes. How many planes pass through the line?

Remark: Axiom 7 is the basis for the common phrase "three points determine a plane."

A8. If two planes intersect, then their intersection is a line.

This is illustrated by our work on the axiom preceding. You can see this in the corner of a room where two walls come together. Is there more that one intersection between the two walls?

Points, Lines, Distances and Midpoints

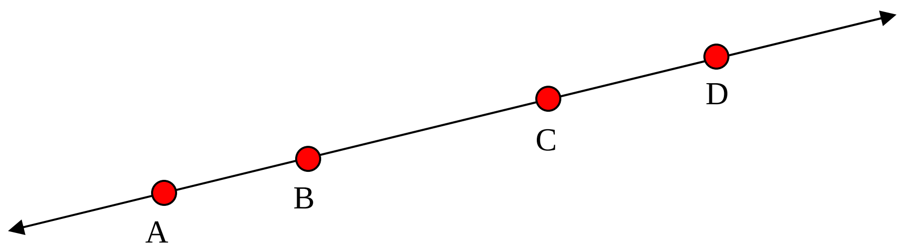
We know from axiom 1 that if A and B are distinct points, then there is exactly one line that passes through both A and B. We introduce some notation below so that we can refer to this line.

Definition: If A and B are distinct points then $\overleftrightarrow{AB}$ denotes the line that passes through both A and B.

If A and B are distinct points on a line, then Axiom 3 allows us to assign real numbers as coordinates to these points. Once this is done, every real number between these numbers corresponds to points on the line $\overleftrightarrow{AB}$. As a result, there are infinitely many points on the line $\overleftrightarrow{AB}$, we can use the definition above to rename the line $\overleftrightarrow{AB}$ in terms of any distinct pair of these points. We summarize this information below.

Theorem: A line contains infinitely many points, and it can be named for any two points that are on the line.

Example: If A, B, C, and D are all points on the same line the line may be called $\overleftrightarrow{AB}$ or $\overleftrightarrow{CD}$, and both names refer to the same line.



Exercise: How many names can be given for the line above? Think carefully about your answer. Are there more points on the line than A, B, C and D?

Exercise: How many names can be given for the line above if we only use the points A, B, C and D to name the line?

If points are given in the Cartesian plane, we can use the distance formula to determine the distance between the points. Recall that if A and B are points in the Cartesian plane with A = (x_1, y_1) and B = (x_2, y_2), then the distance formula gives

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Exercise: Suppose A = (–1, 2) and B = (1, 6). Find AB.

If A and B are points in the Cartesian plane with $\mathrm{A} = (x_1, y_1)$ and point $\mathrm{B} = (x_2, y_2)$, where x_1, y_1, x_2 and y_2 are integers, then AB will typically be an irrational number. There are some exceptions. For example, if A = (0,0) and B = (3,4) then

$$AB = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

However, more typically, the result will be irrational.

We need some notation to discuss points that lie between A and B on the line $\overleftrightarrow{\mathrm{AB}}$.

<u>Definition</u>: Let A and B be distinct points. If C is any point, we say that the point C lies between A and B, denoted A – C – B, if and only if AC + CB = AB.

We can also give a theorem based upon this definition and our axioms.

<u>Theorem</u>: Let A, B and C be distinct points. The points A, B and C are collinear if and only if either A – C – B, C – A – B, or A – B – C.

<u>Exercise</u>: Let A = (–2, 0), B = (5, 0), and C = (1, 1). Is A – C – B? Are these points collinear? If so, which of A – C – B, C – A – B and A – B – C is true?

We also need some notation for line segments and arrays.

<u>Definition</u>: Let A and B be distinct points. The **line segment** $\overline{\mathrm{AB}}$ consists of all points C such that A – C – B. The **ray** $\overrightarrow{\mathrm{AB}}$ consists of all points D such that A – B – D.

<u>Remark</u>: If A and B are distinct points, then $\overleftarrow{\mathrm{AB}}$ will sometimes be used to denote $\overrightarrow{\mathrm{BA}}$. Also, sometimes a **ray** is referred to as a **half-line**.

<u>Theorem</u>: (Midpoints) Let A and B be distinct points. There is exactly one point C such that A – C – B and AC = CB.

Proof: We are given that A and B are distinct points. By definition, $\overleftrightarrow{\mathrm{AB}}$ is the line through the points A and B. From Axiom 3, we can assign the number 0 to the point A and the number AB to the point B. Also, from Axiom 3, there is a point C on the line $\overleftrightarrow{\mathrm{AB}}$ associated with the number AB/2. AC = CB since

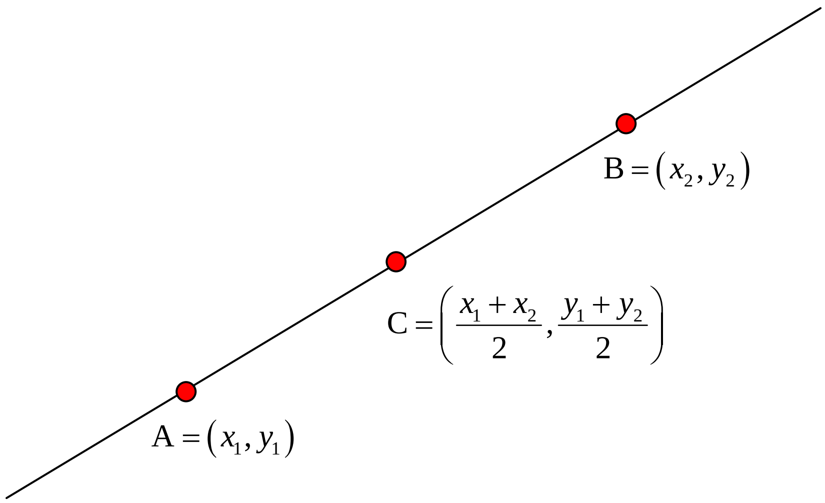
$$\mathrm{AC} = \left|\mathrm{AB}/2 - 0\right| = \mathrm{AB}/2 \text{ and } \mathrm{CB} = \left|\mathrm{AB} - \mathrm{AB}/2\right| = \mathrm{AB}/2$$

Consequently, by definition, A – C – B. The uniqueness of C follows from Axiom 3. •

<u>Remark</u>: This point C guaranteed by the theorem above is called the **midpoint** of the line segment $\overline{\mathrm{AB}}$.

When the points A and B are given in the Cartesian plane, we can use the midpoint formula to find the midpoint of $\overline{AB}$. In general, if the points are point $A = (x_1, y_1)$ and point $B = (x_2, y_2)$, then the midpoint C is given by

$$C = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



Exercise: Let A = (−1, 2) and B = (1, 6). Show that AB = $2\sqrt{5}$, and find the coordinates of the midpoint C of the line segment $\overline{AB}$.

Geometric Coordinates

A close reading of axiom 3 guarantees that in pure geometry we can have one number that specifies a point on a line, instead of two. At first appearance, it looks as though this allows us to specify a point with less information than we use in the Cartesian plane. But this is not correct since a complete description of the point will still require knowledge of the line.

We illustrate a method below for assigning numbers to points on a given line in the manner given in axiom 3. These numbers are referred to as **geometric coordinates** for the points, as opposed to Cartesian coordinates for the points.

This process is simple for a vertical line in the Cartesian plane. Every vertical line can be described by an equation of the form $x = c$ for some given number *c*. Then the Cartesian coordinates for the points on this line will all have the form (c, y), where *y* is an arbitrary real number. In this case, the number *y* can be used a **geometric coordinate** for the point (c, y) on the line.

A non-vertical line in the Cartesian plane can be described by an equation of the form

$$y = mx + b$$

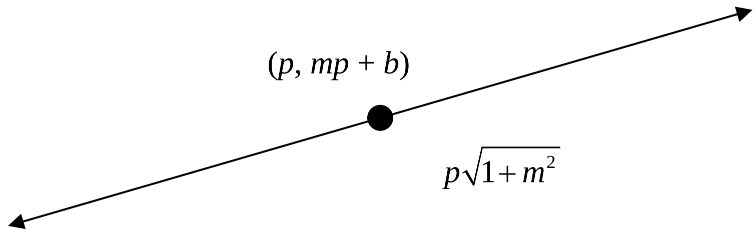
where *m* is the slope of the line and *b* is the *y* intercept. Let P and Q be points on the line $y = mx + b$. Then we can write these points using Cartesian coordinates in the form P = (*p*, *r*) and Q = (*q*, *s*). From the equation of the line, $r = mp + b$ and $s = mq + b$, so

$$\begin{aligned} \text{PQ} &= \sqrt{(q-p)^2 + (s-r)^2} \\ &= \sqrt{(q-p)^2 + ((mq+b)-(mp+b))^2} \\ &= \sqrt{(q-p)^2 + (mq-mp)^2} \\ &= |q-p|\sqrt{1+m^2} \end{aligned}$$

Notice that the final expression above can be written in the form

$$|q-p|\sqrt{1+m^2} = \left|q\sqrt{1+m^2} - p\sqrt{1+m^2}\right|$$

Consequently, we can assign a **geometric coordinate** to a point on this line by multiplying the *x* coordinate of the point by the factor $\sqrt{1+\text{m}^2}$. This gives the **geometric coordinate** of P as $p\sqrt{1+m^2}$, and the **geometric coordinate** of Q as $p\sqrt{1+m^2}$. From the calculations above, the absolute value of the difference of these values is the distance from P to Q. The line is shown below along with both the Cartesian coordinates and the geometric coordinates for an arbitrary point.



The Cartesian coordinates are (p, $mp + b$)
and the geometric coordinate is $p\sqrt{1+m^2}$.

Exercise: Use the method above to give the geometric coordinates for the point A = (−1, 2) and B = (1, 6) on the line $\overleftrightarrow{AB}$. Then, verify that the absolute value of the difference of these coordinates gives the same value for AB as the distance formula.

Exercise: Consider the line $y = 3x + 1$. Verify that (1, 4) and (3, 10) are on this line. Then, use the method above to give geometric coordinates for the points (1, 4) and (3, 10). Finally, verify that the distance given by the distance formula is the same as the absolute value of the difference of the geometric coordinates.

Exercise: The discussion prior to the exercises above showed that the value $p\sqrt{1+m^2}$ could be assigned as a geometric coordinate for any point P = (p, $mp + b$) on the line $y = mx + b$. Show that if c is any fixed real number, then the value $p\sqrt{1+m^2} + c$ can be assigned to each point P = (p, $mp + b$) on this line, and these values can also be used as geometric coordinates!

Remark: The value c in the exercise above can be chosen to make the geometric coordinate 0 for a specific point on a given line.

Definition: Two segments are said to be congruent if they have the same length.

Exercise: Are the two line segments defined by the following endpoints congruent?

a. Segment A has endpoints (−5, 2) and (7, −2).
b. Segment B has endpoints (7, 2) and (5, −2).

What do these segments look like when they are graphed in the Cartesian plane?

Angles and Perpendicular Lines

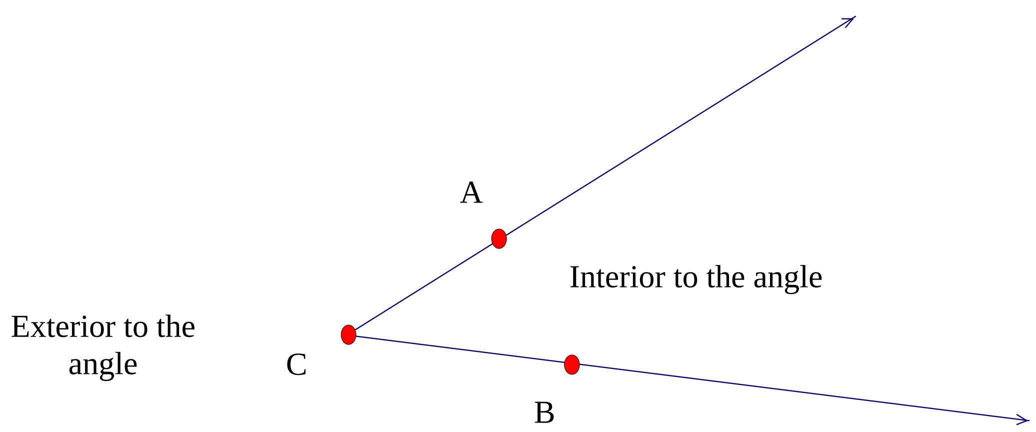
Definition: An angle is a shape created by two rays that share a common endpoint called the **vertex of the angle**. Each ray is called a **side of the angle**.

Definition: A point P is said to be an **interior to an angle** if and only if there is a line that intersects the two rays of the angle in exactly two points A and B, and A – P – B.

Exercise: Explain why there is no point interior to an angle whose rays are collinear.

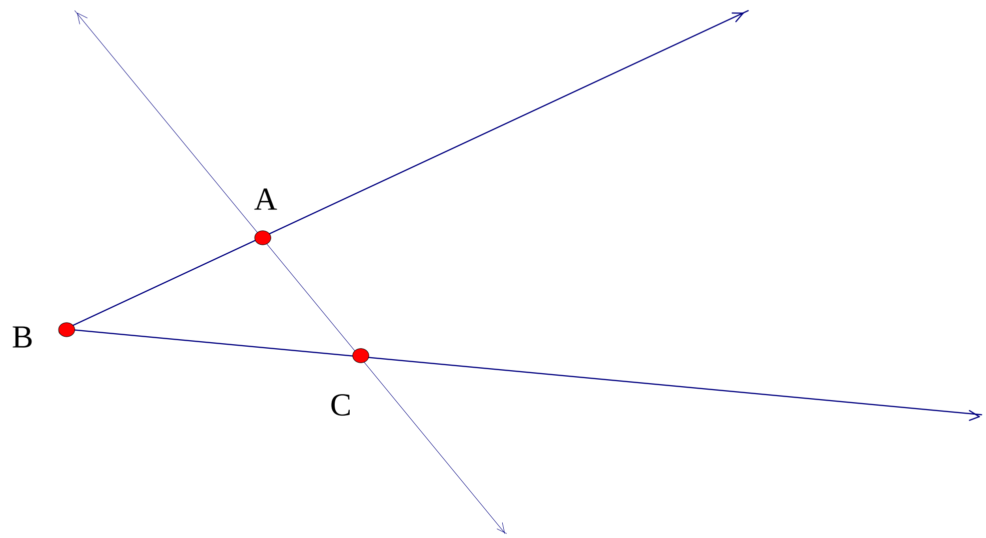
Definition: Let A, B and C be distinct points. An angle is created from the rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is referred to as angle ACB, or simply $\angle ACB$.

If the rays that define the angle are not collinear, then the angle divides the plane into three sets of points. One set consists of the **vertex of the angle** and the points on the rays and that make up the angle. The second set consists of points that are **interior to the angle**. The third set consists of the compliment of the union of the first two sets, and it is referred to as the set of points **exterior to the angle**. Note, that since an angle measure cannot equal 0° nor 180° we have no ambiguity about how to find an angle's interior.



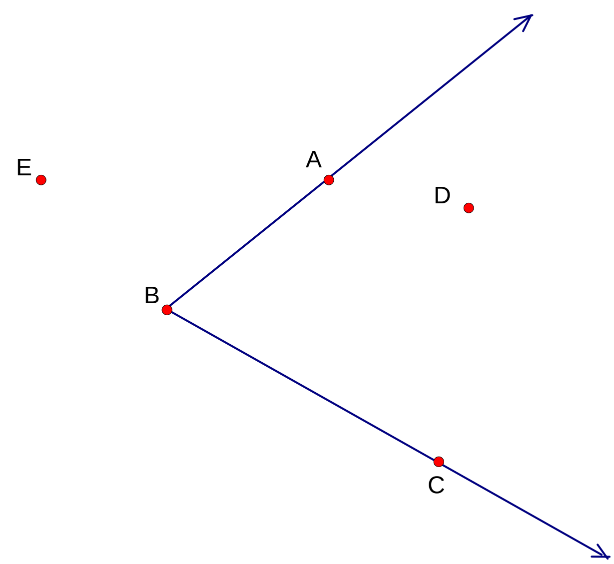
The figure above might help visualize the interior and exterior of an angle. Another method for visualizing the interior of an angle involves a line that intersects the rays of the angle.

Definition: Any line that intersects two other lines or parts of lines is called a **transversal** of the lines, rays, or segments intersected.



Exercise: Angle ABC, denoted $\angle$ ABC, is shown with transversal $\overleftrightarrow{AC}$ in the figure above. Which parts of $\overleftrightarrow{AC}$ are interior to $\angle$ ABC and which are exterior?

Exercise: Consider $\angle$ ABC in the figure below. We know $\angle$ ABC divides the plane ABC into 3 sets of points. Give the location of the points A, B, C, D and E with respect to these three point sets.



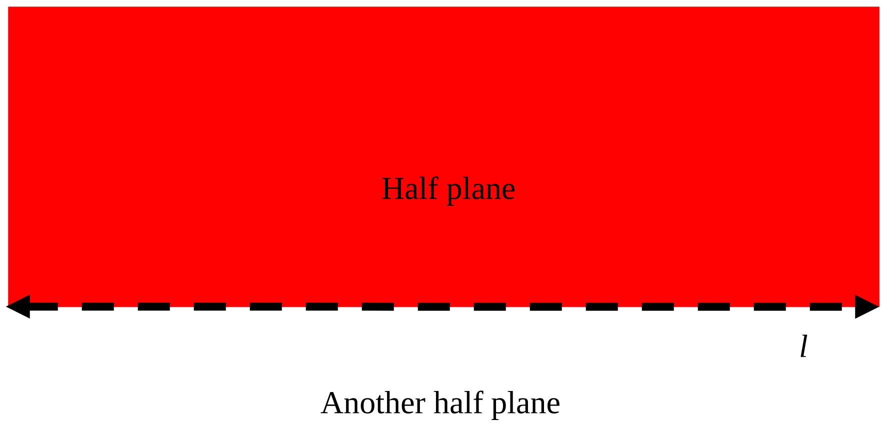
We need a separation axiom to help define the measure of an angle.

A9. **(The Plane Separation Postulate)** Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets, such that if P is in one set and Q is in the other, then $\overline{PQ}$ intersects the line.

The strength of the axiom above is that it allows us to define a **half plane**.

Definition: A set of points *S* in a plane is said to be a **half plane** if and only if there is a line *l* in the plane such that *S* is one of the sets referred to in the axiom above with respect to *l*. The **edge of the half plane** is the line *l*.

The idea is illustrated in the figure below. Of course, the line *l* does not have to be horizontal.



Now we continue with the angle axioms.

A10. **(The Angle Measurement Postulate)** To every angle there corresponds a real number *r* between 0 and 180 referred to as the degree measure of the angle.

When an angle is measured in this way, the number corresponding to the measure of the angle is appended with a super script ^{o}. For example, we write 30^{o} if the value *r* referred to above is 30.

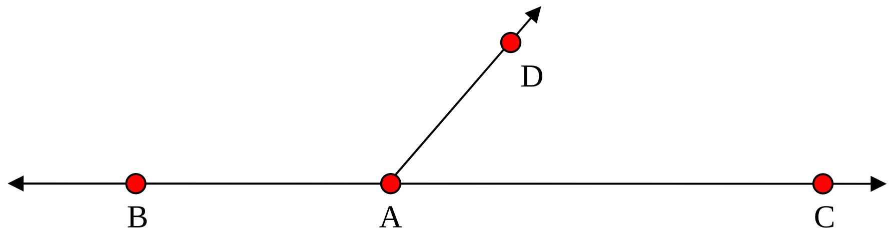
If P, A and B are distinct points, the **measure of $\angle$ PAB** is denoted by m $\angle$ PAB.

A11. **(The Angle Construction Postulate)** Let $\overrightarrow{AB}$ be a ray on the edge of the half plane H. For every r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$ with P in H such that m $\angle$ PAB = r^{o}.

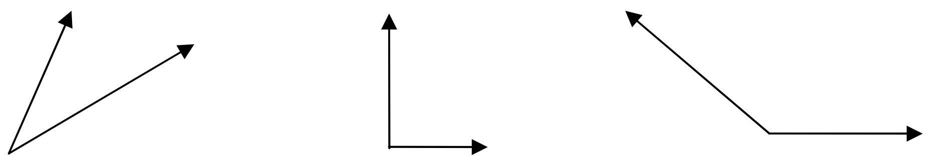
A12. **(The Angle Addition Postulate)** If D is a point in the interior of $\angle$ BAC, then m $\angle$ BAC = m $\angle$ BAD + m $\angle$ DAC.

Definition: Two angles are said to form a **linear pair** if and only if they are formed by three distinct rays, and two of the rays are collinear.

$\angle$ BAD and m $\angle$ DAC form a **linear pair** in the figure below. Notice that these angles are formed by the three distinct rays $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AC}$, and the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear. The fact that the rays are distinct implies that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form the line $\overleftrightarrow{AC}$.

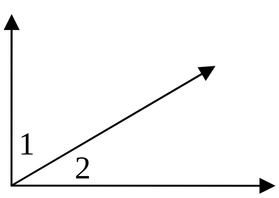


Definition: An angle with a measure less than 90° is called an **acute angle**. An angle with a measure of 90° is called a **right angle**. An angle with a measure greater than 90° is called an **obtuse angle**.

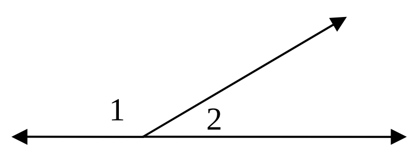


acute angle   right angle   obtuse angle

Definition: Two angles whose measures sum to 90° are called **complementary angles**. Two angles whose measure sum to 180° are called **supplementary angles**.



$\angle 1$ and $\angle 2$ are complementary



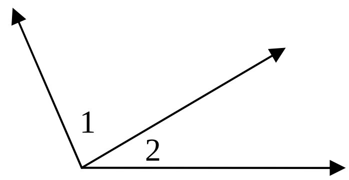
$\angle 1$ and $\angle 2$ are supplementary

Remark: When you list the sum of the angle measures 90° and 180° in numerical order, the names complementary and supplementary are in alphabetical order.

A13. **(The Supplement Postulate)** If two angles form a linear pair, then they are supplementary.

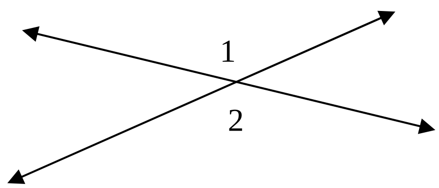
Definition: Two angles, A and B, with the same measure are called **congruent angles**. We use the notation $\angle A \cong \angle B$ for the statement *A and B are congruent angles*.

Definition: **Adjacent angles** are a pair of angles with a common vertex, a common side, and no interior points in common.



$\angle 1$ and $\angle 2$ are adjacent angles

Definition: Two angles that are not adjacent and yet are formed by the intersection of two lines are called **vertical angles**.



$\angle 1$ and $\angle 2$ are vertical angles

Theorem: Two angles that are either complementary or supplementary to the same angle are congruent.

Proof: (complementary portion) Let $\angle ABC$ and $\angle STV$ be complementary to $\angle XYZ$. From the definition of complementary angles

$$m\angle ABC + m\angle XYZ = 90°$$
$$m\angle STV + m\angle XYZ = 90°$$

Subtracting these equations gives the equivalent equation

$$m\angle ABC - m\angle STV = 0$$

Therefore, $m\angle ABC = m\angle STV$. •

Exercise: Prove the *supplementary* portion of the theorem above. Hint: Modify the proof given for the "complementary" portion.

Theorem: Two angles that are equal in measure and are supplementary angles are right angles.

Proof: Let $\angle ABC$ and $\angle STV$ be supplementary angles, and suppose

$$\text{E1:} \quad m\angle ABC = m\angle STV$$

From the definition of supplementary angles

$$\text{E2:} \quad m\angle ABC + m\angle STV = 180°$$

Substituting E1 into E2 gives the equivalent equation

$$m\angle STV + m\angle STV = 180°$$

Therefore,

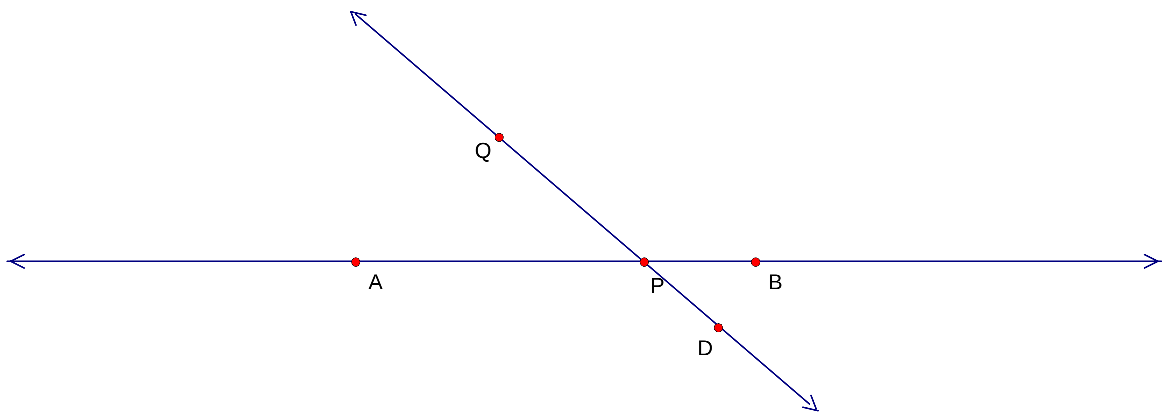
$$m\angle STV = 90°, \text{ so } \angle STV \text{ is a right angle.}$$

From E1 and the measure of $\angle$ STV, we know

$$m\angle ABC = 90°, \text{ so } \angle ABC \text{ is also a right angle.} \bullet$$

<u>Theorem</u>: Vertical angles are congruent.

Proof: Consider the figure below.



By definition, $\angle$QPA and $\angle$ BPD are vertical angles. Also, $\angle$ BPD and $\angle$ QPB are linear pairs. Therefore, from axiom 13,

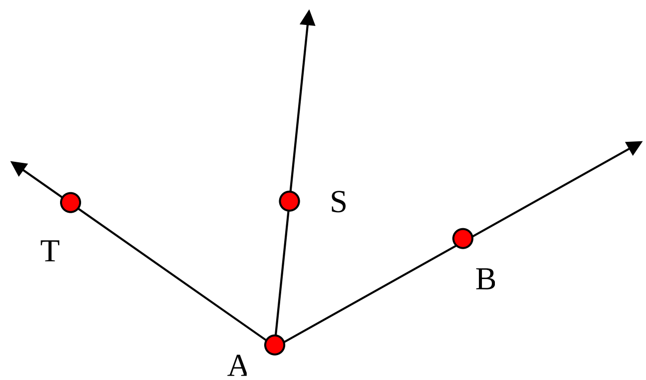
$$\text{E1} \quad m\angle QPA + m\angle QPB = 180°$$
$$\text{E2} \quad m\angle QPB + m\angle BPD = 180°$$

Subtracting E2 from E1 gives $m\angle QPA = m\angle BPD$, implying $\angle QPA \cong \angle BPD$. •

Notice that a special case can occur with vertical angles if one of the angles is a right angle. In this case, from the theorem above, the angle vertical to this angle will be a right angle, and the angles adjacent to these angles will be right angles. That is, **all four angles formed by the intersection of two lines will be right angles if and only if one of the angles is a right angle**. This motivates the following definition.

Definition: Two lines are **perpendicular** if and only if one of their angles of intersection is a right angle. Two line segments or rays are **perpendicular** if their linear extensions are perpendicular.

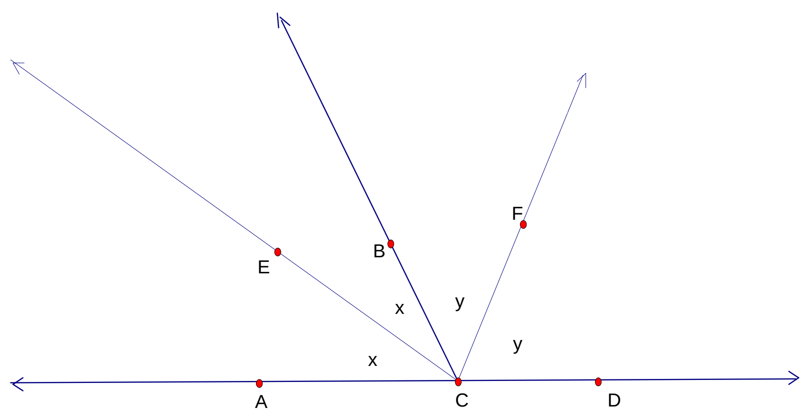
Definition: Any ray, segment, or line that divides an angle into two congruent angles is called an **angle bisector**.



$\overrightarrow{AS}$ is the **angle bisector** of ∠ TAB.

Theorem: The angle bisectors of a linear pair are perpendicular.

Proof: Consider the figure below.



∠ACB and ∠ BCD are a linear pair. By axiom 13, we know

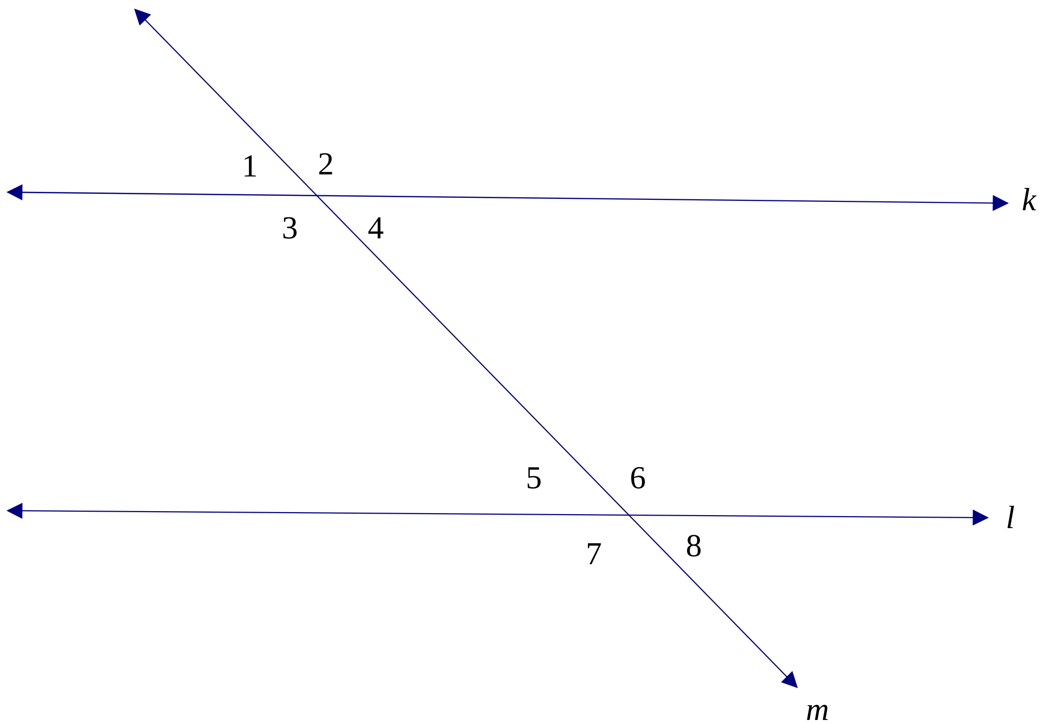
$$m\angle ACB + m\angle BCD = 180°$$

The angle bisector of ∠ ACB makes two angles having measure ½ m ∠ ACB. We call this value x. So, m∠ ACB = 2x. Similarly the angle bisector of ∠ BCD makes two congruent angles having measure ½ m ∠ BCD. We call this value y. So, m∠ BCD = 2y.

Using substitution, we have $2x + 2y = 180°$, which is equivalent to $x + y = 90°$. Consequently, ∠ ECF is a right angle, and therefore $\overrightarrow{CE}$ and $\overrightarrow{CF}$ are perpendicular. •

Parallel Lines

We need some terminology to discuss parallel lines. The lines *k* and *l* in the figure below have been drawn to appear parallel, but the terms in the definition that follow do not depend upon the lines being parallel. The line *m* is a transversal of *k* and *l*.



Definition: ∠ 1 and ∠ 5 are **corresponding angles**. As are ∠ 3 and ∠ 7, ∠ 2 and ∠ 6, and ∠ 4 and ∠ 8. ∠ 3, ∠ 4, ∠ 5 and ∠ 6 are **interior angles**. ∠ 1, ∠ 2, ∠ 7 and ∠ 8 are **exterior angles**. ∠ 3 and ∠ 5 are **interior angles on the same side of *m*,** as are ∠ 4 and ∠ 6. ∠ 1 and ∠ 7 are **exterior angles on the same side of *m***, as are ∠ 2 and ∠ 8. ∠ 3 and ∠ 6 are **alternate interior angles**, as are ∠ 4 and ∠ 5. ∠ 1 and ∠ 8 are **alternate exterior angles**, as are ∠ 2 and ∠ 7.

We can use the language developed above to give a definition of parallel lines.

Definition: Two distinct lines *k* and *l* are parallel if and only if whenever they are cut by a transversal *m*, the interior angles on the same side of *m* are supplementary.

A14. **(The Parallel Postulate)** If *l* is a line and P is a point not on *l*, then there is one and only one line contained in the plane of P and *l* that passes through P and is parallel to *l*.

The definitions and axioms above have many consequences. We state some of these results below.

Theorem: Two lines are parallel if and only if any of the following are true:

1. A pair of alternate interior angles around a transversal are congruent.
2. A pair of alternate exterior angles around a transversal are congruent.
3. A pair of corresponding angles around a transversal are congruent.

Proof: (#1) Consider the lines *k* and *l* cut by the transversal *m* in the figure above. Notice that $\angle$ 3 and $\angle$ 6 are alternate interior angles. By definition, *k* and *l* are parallel if and only if $\angle$ 3 and $\angle$ 5 are supplementary. We know that $\angle$ 5 and $\angle$ 6 are supplementary. Consequently,

$$m \angle 3 + m \angle 5 = 180$$
$$m \angle 5 + m \angle 6 = 180$$

Subtracting gives

$$m \angle 3 = m \angle 6$$

Therefore, *k* and *l* are parallel if and only if $\angle 3 \cong \angle 6$. •

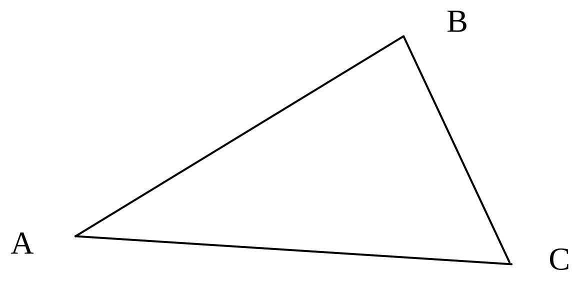
Exercise: Prove #2 in the theorem above. Hint: Use # 1 and the fact that vertical angles are congruent.

Exercise: Prove # 3 in the theorem above. Hint: Use either #1 or #2 and supplementary angles.

The results above have important consequences for triangles.

Definition: A **triangle** is a closed figure in a plane consisting of three segments called sides, with any two sides intersecting in exactly one point called a vertex.

Triangles are typically named for their vertices. The triangle ABC is shown below.

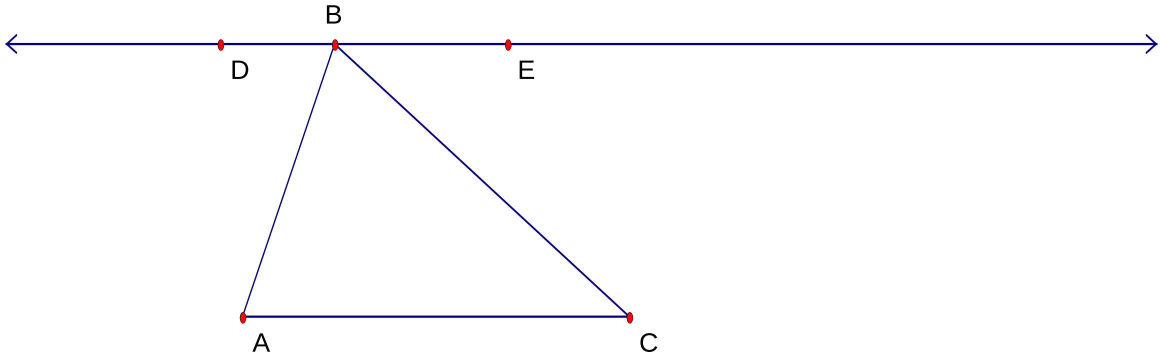


Angles BAC, ACB and CBA are the **interior angles of triangle ABC**. In this setting, $\angle BAC$ is commonly referred to as $\angle A$. Similarly, $\angle ACB$ is commonly referred to as $\angle C$ and $\angle CBA$ is commonly referred to as $\angle C$. The following result tells us that

$$m \angle A + m \angle B + m\angle C = 180^{o}.$$

<u>Theorem</u>: The sum of the measures of the interior angles of a triangle is 180^o.

Proof: Let triangle ABC be any triangle. Construct a line through B parallel to $\overleftrightarrow{AC}$ with two points on it D and E such that D – B – E .



$\angle$ DBA and $\angle$ ABE are a linear pair so they are supplementary. As a result,

$$m\angle DBA + m\angle ABE = 180°$$

The point C is in the interior of $\angle$ ABE, so

$$m\angle ABE = m\angle ABC + m\angle CBE.$$

We can substitute $m\angle ABC + m\angle CBE$ for $m\angle ABE$ in the equation above. This gives

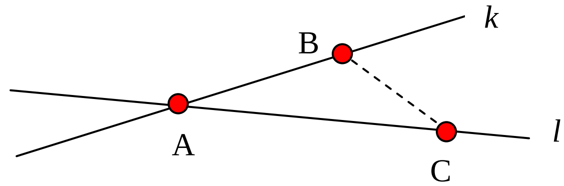
$$m\angle DBA + m\angle ABC + m\angle CBE = 180°$$

Also, $\angle$ DBA and $\angle$ BAC are alternate interior angles for the parallel lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BE}$, and $\overleftrightarrow{AB}$ is a transversal cutting these lines. Consequently, $m\angle DBA = m\angle BAC$. Similarly, $m\angle CBE = m\angle BCA$. Substituting this information above gives

$$m\angle BAC + m\angle ABC + m\angle BCA = 180° \bullet$$

<u>Theorem</u>: Parallel lines do not intersect.

Proof: Consider the figure below with the distinct lines *k* and *l* intersecting at the point A. The points B and C lie on the lines *k* and *l* respectively.

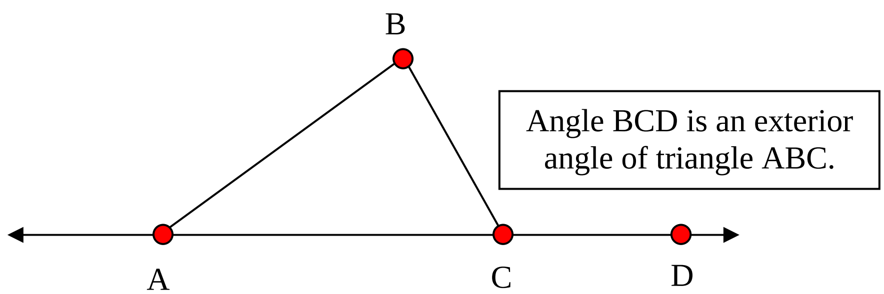


Since the lines *k* and *l* are distinct, $m\angle BAC > 0$. Also, from the theorem above,

$$m\angle BAC + m\angle ABC + m\angle BCA = 180°$$

Therefore, angle ABC and angle BCA are not supplementary. So by definition, *k* and *l* are not parallel lines. •

We conclude this section with the notion of the **exterior angle** theorem for triangles. Let ABC be a triangle and extend the line through A and B. Choose a point D on the line $\overleftrightarrow{AC}$ so that A – C – D. ∠ BCD is referred to as an **exterior angle of triangle ABC**.



∠BCA is referred to as the **adjacent interior angle** to the exterior angle BCD, and ∠BAC and ∠ABC are referred to as **remote interior angles** to the exterior angle BCD.

<u>Exercise</u>: How many exterior angles are there for a triangle?

<u>Theorem</u>: The measure of an exterior angle of a triangle is equal to the sum of the measures of the corresponding remote interior angles of the triangle.

Proof: Note that an exterior angle and its adjacent interior angle are a linear pair so the sum of their measures is 180°. From the theorem immediately preceding we know that the sum of the interior angles of the triangle is also 180°. Thus:

$$m\angle ABC + m\angle BCA + m\angle BAC = m\angle BCA + m\angle BCD$$

Subtracting m ∠ BCA from both sides we find

$$m\angle ABC + m\angle BAC = m\angle BCD \bullet$$

<u>Exercise</u>: Determine the sum of the exterior angles of a triangle.