Shadows on the Wall

Purpose:
Participants will apply the properties of a dilation to solve a problem. They will extend the use of these properties in a coordinate plane to solve a problem that integrates algebraic concepts.

Overview.
Participants will solve a problem that involves a light source and a polygon that has been projected onto a wall. They will use properties of a dilation to analyze and solve the problem. Properties of a dilation will also be applied to a problem situation in the coordinate plane in order to determine a solution using algebraic concepts.

TExES Mathematics 4-8 Competencies. The beginning teacher:

III.008.A Selects and uses appropriate units of measurement (e.g., temperature, money, mass, weight, area, capacity, density, percents, speed, acceleration) to quantify, compare, and communicate information.
III.009.A Understands concepts and properties of points, lines, planes, angles, lengths, and distances.
III.009.B Analyzes and applies the properties of parallel and perpendicular lines.
III.010.B Applies relationships among similar figures, scale, and proportion and analyzes how changes in scale affect area and volume measurements.
III.010.C Uses a variety of representations (e.g., numeric, verbal, graphic, symbolic) to analyze and solve problems involving two-and three-dimensional representations (e.g., projections, cross-sections, nets) and uses these representations to solve problems.
III.011.C Uses dilations (expansions and contractions) to illustrate similar figures and proportionality.
III.011.E Applies concepts and properties of slope, midpoint, parallelism, and distance in the coordinate plane to explore properties of geometric figures and solve problems.
III.011.F Applies transformations in the coordinate plane.
V.016.C Expresses mathematical statements using developmentally appropriate language, standard English, mathematical language, and symbolic mathematics.
V.016.D Communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphic, pictorial, symbolic, concrete).

TEKS Mathematics Objectives. The student is expected to:

4.8A Identify right, acute, and obtuse angles.
4.8B Identify models of parallel and perpendicular lines.
5.7A Identify critical attributes including parallel, perpendicular, and congruent parts of geometric shapes and solids.
5.7B Use critical attributes to define geometric shapes or solids.
5.9 Locate and name points on a coordinate grid using ordered pairs of whole numbers.
6.3A Use ratios to describe proportional situations.
6.4A Use tables and symbols to represent and describe proportional and other relationships involving conversions, sequences, perimeter, area, etc.
6.6A Use angle measurements to classify angles as acute, obtuse, or right.
6.7 Locate and name points on a coordinate plane using ordered pairs of non-negative rational numbers.
6.8B Select and use appropriate units, tools, or formulas to measure and to solve problems involving length (including perimeter and circumference), area, time, temperature, capacity, and weight.
7.6D Use critical attributes to define similarity.
7.7 Locate and name points on a coordinate plane using ordered pairs of integers.
8.2D Use multiplication by a constant factor (unit rate) to represent proportional relationships.

8.3B Estimate and find solutions to application problems involving percents and proportional relationships such as similarity and rates.
8.4 Generate a different representation of data such as a table, graph, equation, or verbal description.
8.5 Generate similar shapes using dilations including enlargements and reductions.
8.7D Locate and name points on a coordinate plane using ordered pairs of rational numbers.

Terms.
Dilation, center of dilation, enlargement, reduction, proportional, ratio, similar figures, corresponding sides/angles, constant of proportionality, constant ratio, perimeter, area, scale factor, coordinates, coordinate plane

Materials.

- Grid paper
- Straight edge
- Calculator
- Patty paper or tracing paper

Transparencies.

- *Shadows on the Wall*

Activity Sheet(s).

- *Shadows on the Wall*

Procedure.

Steps	Questions/Math Notes
1. Have participants read the problem on Transparency *Shadows on the Wall* . Ask them to study the diagram on the Activity Sheet *Shadows on the Wall* . Have participants work on the problem individually for about 5 minutes. After reflecting on the problem, have them share their ideas with a partner and continue to work in pairs on a solution. Monitor the participants' work and ask probing questions to help them clarify and extend their thinking about dilations.	Ask participants questions about their understanding of the problem and properties of a dilation. *What does the light source represent in this problem?* *How would you describe this transformation?* *What are some properties of this transformation?* *What stays the same in the two trapezoids? What changes? How do you know?* *What is significant about the distances of two corresponding vertices from the given light source?*
2. Have several groups with different and unique strategies share with the whole group on the overhead. Ask them to justify their statements using definitions, axioms, and/or theorems from geometry.	*How did you determine a scale factor in this problem?* *How did you use this scale factor in the solution of the problem?*

	If the scale factor is greater than 1,what type of change do you expect from figure A to figure B? Explain. *If you want to reduce the size of a geometric figure, what scale factor would you use and why?* *How did you use ratios to help you solve this problem?* *How would a change in the position of the light source affect the size of the image?* *How did you determine the dimensions of the rectangular piece of plywood for the table top?* *How did you determine how to make two cuts on the rectangular piece of plywood to form trapezoid A'B'C'D'?* *What are other way(s) to make the two cuts?*

Possible solution: A'B' = D'C' = 2 ft., A'D' = 3 ft., and B'C' = 1.5 ft.
The rectangular section of plywood should be cut to a length of 3 feet and a height of approximately 1 foot 11 inches (rounded to the nearest inch) .

The diagram of the problem situation represents a dilation with the light source as the center of dilation. Trapezoid A'B'C'D' is similar to trapezoid ABCD because a dilation is a similarity transformation. Therefore, corresponding angles are congruent and corresponding sides are in proportion. By definition of a dilation, if A is mapped onto A' and D is mapped onto D' by a scale factor of "k", then A'D' = k AD. It follows that A'B' = k AB, B'C' = k BC, and C'D' = k CD. Using the measurements given in the problem, a possible solution follows.

OA'/ OA = OB'/ OB = OC'/ OC = OD'/ OD = k (property of a dilation)

A'D'/ AD = A'B'/ AB = B'C'/ BC = C'D'/ CD = k (corresponding sides of similar figures are in proportion)

OA = 18 in. = 1.5 ft. (measurement conversion)

OA' = OA + AA' (definition of between or segment addition)
OA' = 1.5 ft + 7.5 ft. = 9 ft. (substitution and addition)

OA'/ OA = 9/1.5 = 6/1 = k (property of a dilation)

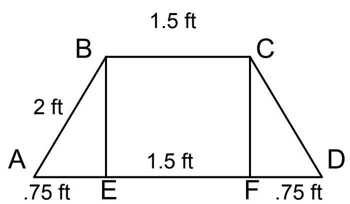
AB = 4 in. = 4/12 ft. = 1/3 ft. (measurement conversion)

BC = ¾ AB = ¾ x 1/3 = 1/4 ft. (substitution and multiplication)

BC: AD = 1: 2 → AD is twice BC (interpretation of a ratio)
AD = 2 x 1/4 = 1/2 ft. (substitution and multiplication)

A'B'/AB = 6/1 → A'B'/1/3 = 6/1 → A'B' = 2 ft. (The ratio of corresponding sides has the same value as the scale factor, k.)

A'D'/AD = 6/1 → A'D'/1/2 = 6/1 → A'D' = 3 ft.
B'C'/BC = 6/1 → B'C'/ ¼ = 6/1 → B'C' = 1.5 ft.
A'B' = C'D' = 2 ft.



Segments BE and CF are drawn perpendicular to segment AD from points B and C, respectively. Triangle AEB is a right triangle with the lengths of two sides given. Therefore, the Pythagorean theorem applies and $2^2 = (BE)^2 + (0.75)^2$. It follows that $4 = (BE)^2 + 0.5625$, $(BE)^2 = 3.4375$, and BE ≈ 1.9 feet rounded to the nearest tenth of a foot. 9/10 times 12 inches is approximately 10.8 inches or 11 inches rounded to the nearest inch. BE can be expressed as 1 foot 11 inches for this project.

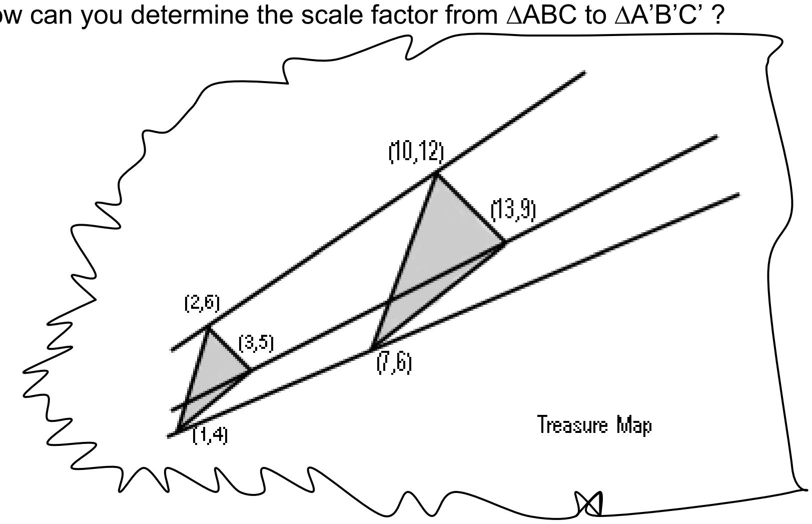
Using this information, a rectangular sheet of plywood would need to be cut the length of the larger base of the trapezoid (3 ft) and a height of BE (1 ft 11 in). You would only need to make 2 cuts along AB and CD to make the table top. Using right triangle trigonometry, cos A = 0.75/2 and the degree measure of angle A is 37.5°. This angle measure could help you determine AB and CD.

Extension:

A partially torn map was discovered in an old attic. Two triangles were drawn on a coordinate plane with vertices labeled A,B,C on one triangle and A'B'C' on the other. Points A', B', and C' represent the locations of three tall trees ; points A, B, and C represent the locations of three large boulders. The map stated, "The treasure is buried at the center of dilation of these two triangles." .

a. Explain how to find the coordinates of the point that marks the spot where the treasure is buried.

b. How can you determine the scale factor from ΔABC to $\Delta A'B'C'$?



Solution using the TI-83+ graphing calculator:

a. Press STAT and ENTER. Clear lists 1 and 2. Enter the x-coordinates of points C and C" into L_1 and the corresponding y-coordinates into L_2. (Ask your instructor for assistance if needed.) Press STAT, CALC, 4, ENTER, ENTER. The screen will show y=ax + b with values for "a" and "b". Substitute the values for "a" and "b" into the equation to get Y= 0.75x + 4.5. Press Y= and enter this equation in y_1. This is the equation of the line containing the points C and C'. This line also contains the center of dilation. Next,enter the data from points B and B' into lists 3 and 4. Press STAT, CALC, 4, 2^{nd} 3, comma, 2^{nd} 4, ENTER. Use the values for "a" and "b" to write a second equation, y = 0.4x + 3.8. Enter this equation into y_2. This equation represents the line that contains points B , B', and the center of dilation. Set an appropriate window or press ZOOM 6 to get a standard viewing rectangle. Press GRAPH to graph the equations of the two lines. The point of intersection of these two lines represents the center of dilation and the spot where the treasure is buried. To find this point of intersection, press 2^{nd} CALC, 5, and "make three wishes" by pressing ENTER (I wish.), ENTER (I wish.), ENTER (I wish I knew where that treasure is buried!). Your "wish" will come true on the screen with the coordinates (-2,3).

b. Use the distance formula to find B'C' and BC as follows:

$B'C' = \sqrt{((3-2)^2 + (5-6)^2)} = \sqrt{(1+1)} = \sqrt{2}$

$BC = \sqrt{((13-10)^2 + (9-12)^2)} = \sqrt{(9+9)} = 3\sqrt{2}$

$B'C'/BC = 3\sqrt{2} / \sqrt{2} = 3/1$

On the map, k=3 and OC/ OC' = 3/1 where O represents the center of dilation.

The distance from C' to C is twice the distance from O to C'. Extend CC' to the left and find the midpoint M. Use a compass to copy C'M using C' as and endpoint and make an arc intercepting ray CC' at point O.