Set the Sails!

Purpose:
Participants will use graphing technology to investigate reflections, translations, rotations, and sequences of reflections and translations in the coordinate plane.   They will give verbal descriptions of generalizations and use symbols to describe mapping rules for the transformations.

Overview.
Participants will use Cabri Junior™ on a graphing calculator to investigate transformations (reflections, translations, and rotations) in the coordinate plane. They will graph the outline of a sail in the coordinate plane on a graphing calculator and use this figure for reflections about the axes and the line y=x.   Participants will also explore translations and rotations. They will write verbal and symbolic descriptions to generalize their findings.

TExES Mathematics 4-8 Competencies. The beginning teacher:
III.009.A   Understands concepts and properties of points, lines, planes, angles, lengths, and distances.
III.009.B   Analyzes and applies the properties of parallel and perpendicular lines.
III.010.C   Uses a variety of representations (e.g., numeric, verbal, graphic, symbolic) to analyze and solve problems involving two-and three-dimensional figures such as circles, triangles, polygons, cylinders, prisms, and spheres.
III.011.B   Uses translations, reflections, glide-reflections, and rotations to demonstrate congruence and to explore the symmetries of figures.
III.011F   Applies transformations in the coordinate plane.
V.015.C   Demonstrates an understanding of the use of inductive reasoning to make conjectures and deductive methods to evaluate the validity of conjectures.
V.016.C   Expresses mathematical statements using developmentally appropriate language, standard English, mathematical language, and symbolic mathematics.
V.016.D   Communicates mathematical ideas using a variety of representations (e.g., numeric, verbal, graphic, pictorial, symbolic, concrete)
V.016.F   Uses the language of mathematics as a precise means of expressing mathematical ideas.

TEKS Mathematics Objectives. The student is expected to:
4.8A   Identify right, acute, and obtuse angles.
4.8B   Identify models of parallel and perpendicular lines.
4.9C   Use reflections to verify that a shape has symmetry.
5.7A   Identify critical attributes including parallel, perpendicular, and congruent parts of geometric shapes and solids.
5.8A   Sketch the results of translations, rotations, and reflections.
5.9   Locate and name points on a coordinate grid using ordered pairs of whole numbers.
6.6A   Use angle measurements to classify angles as acute, obtuse, or right.
6.7   Locate and name points on a coordinate plane using ordered pairs of non-negative rational numbers.
7.6B   Use properties to classify shapes including triangles, quadrilaterals, pentagons, and circles.
7.7A   Locate and name points on a coordinate plane using ordered pairs of integers.
7.7B   Graph translations on a coordinate plane.
8.6B   Graph dilations, reflections, and translations on a coordinate plane.
8.7D   Locate and name points on a coordinate plane using ordered pairs of rational numbers.

Terms.
Reflection, translation, rotation, isometry, line of symmetry, angle of rotation, center of rotation, image, preimage, coordinate, ordered pair, quadrant, orientation, angle, perpendicular bisector

Materials.
- TI-83+ graphing calculators with Cabri Junior^{TM} software
- Grid paper (optional)
- Straight edge
- Patty paper or tracing paper
- Sharp pencils

Transparencies.
- *Set the Sails!*

Activity Sheet(s).
- *Set the Sails!*

Procedure.

Steps	Questions/Math Notes
1. Have participants read the problem on the Transparency *Set the Sails!* and discuss any questions they may have about using Cabri Junior^{TM} software on the TI-83+ graphing calculator. Refer them to the Texas Instruments website for information about the guide book providing additional assistance with this technology: www.support.education.ti.com Ask the group if anyone has experience with the Cabri Junior^{TM} software on the TI-83+ graphing calculator. Rearrange the groups, if possible, so that there is at least one person with some experience with the software in a smaller group. Have them work in pairs (each with a graphing calculator) on the investigations. Monitor their progress and ask probing questions to check for understanding about the investigations.	Ask participants questions to stimulate and extend their thinking about transformations in the coordinate plane. *What effect do you expect a reflection to have on the coordinates of the preimage? Explain.* *What is the relationship between the x-or y-axis and the corresponding vertices of the preimage and its reflection about each of these axes?* *What do you expect to change when a geometric figure is translated horizontally in the coordinate plane? When a geometric figure is translated vertically in the coordinate plane? Explain.* *What do you expect to stay the same in a horizontal translation in the coordinate plane? In a vertical translation in the coordinate plane? Why?* *How could you use patty paper and grid paper to help you validate your findings about rotations in the coordinate plane?*
2. Debrief the investigations and have willing participants demonstrate their findings on the overhead calculator. Have participants articulate their findings or generalizations verbally and symbolically.	Ask participants to make generalizations from their graphs and tables about the transformation investigations in the coordinate plane. *Which of the transformations (reflection, translation, rotation) represent an isometry? Justify your answer.* *How was the orientation affected by each of these transformations?*

	How did a reflection about the y-axis affect the coordinates of the preimage? About the x-axis? *What do you predict the coordinates of a point (2,3) to be when reflected about a vertical line x=3? About a horizontal line y= -2?* *What effect did a reflection about the line y=x have on the coordinates of the preimage?* *Describe the meaning of $(x,y) \rightarrow (x -1, y + 4)$ in words and model on a coordinate grid.* *If a geometric figure is translated to the left "a" units and then translated "b" units up in a coordinate plane, will order matter if the sequence of translations is reversed? Explain.* *How were the coordinates of the triangle (sail) affected by a 90 degree rotation about the origin in a clockwise direction? By a 180 degree clockwise rotation about the origin?* *What do you predict the effect to be on the coordinates of the triangle (preimage) when the figure is rotated 90 degrees counterclockwise about the origin? 180 degrees counterclockwise about the origin? Validate your thinking.* *How has the use of graphing technology helped to clarify your thinking about transformations in the coordinate plane?* *How could this technology be used in the classroom to teach geometric concepts?*

Solution. Values shown in the tables that follow will vary by participant, but generalizations should be consistent.

Reflection Investigation using the TI-83+ with Cabri Junior™ software:

1. Reflect the sail about the y-axis:

- Press F5 and select **Hide/Show- axes** to draw a set of axes on the screen.
- Press F2 and select **Triangle**. Draw a triangle in the second quadrant.
- Press F5 and select **Alph-Num** to label the triangle ΔABC (figure 1).
- Press F4 and select **Reflection.** Move the cursor to ΔABC and press ENTER. Then move the cursor to the y-axis and press Enter to reflect ΔABC about the y-axis.
- Press F5 and select **Coord. & Eq.** . Move the cursor to each vertex of the triangle and press Enter each time to display the coordinates (figure 1).
- Press F2 and select **Segment.** Connect corresponding vertices of the image and pre-image with segments.
- Press F5 and select **Measure-D. & Length**. Find the length of the parts that make up each segment.
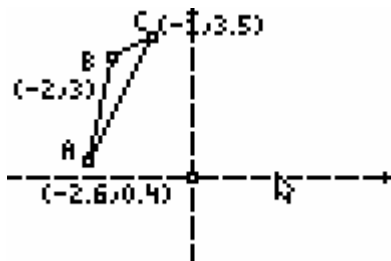
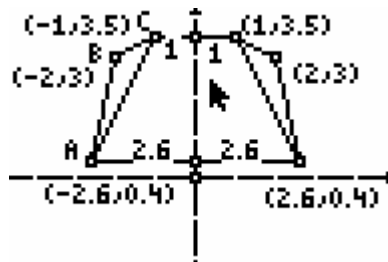Figure 1   Figure 2
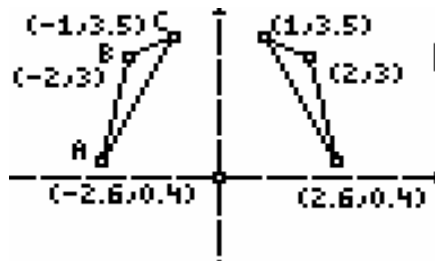
Figure 3

Preimage Coordinates		Image Coordinates	
x	y	x'	y'
-2.6	0.4	2.6	0.4
-2	3	2	3
-1	3.5	1	3.5

a. The x-coordinates of the image are the opposite of the corresponding x-coordinates of the pre-image. The y-coordinates stayed the same. The lengths of segments can be measured to verify that distance remained the same. Corresponding angles can also be measured to verify that angle measure did not change. Therefore, the image and pre-image are congruent. Since angle measure and distance measure remained the same, a reflection is an isometry. However, in figure 2, orientation changed. Triangle ABC is

named in a counterclockwise direction and its image would be named in a clockwise direction.

b. The y-axis is the axis of symmetry in this reflection and is the perpendicular bisector any segment joining a pair of corresponding vertices. Refer to figure 3 to verify that a segment joining a pair of corresponding vertices is bisected. The axes are perpendicular in the figure.

c. A reflection is a transformation that maps each point (x, y) of a preimage in the coordinate plane about a line onto a point (x' , y') of an image such that the line is the perpendicular bisector of the segment joining the corresponding points.

d. $(x, y) \rightarrow (-x, y)$

2. **Reflect the sail about the x-axis:**

- Press F5 and select **Hide/Show-axes** followed by ENTER.
- Press F2 and select **Triangle**. Draw a triangle in the first quadrant and label it △ABC as shown in figure 1.
- Press F5 and select **Coord. & Eq.** to display coordinates of the vertices.
- Press F4 and select **Reflection.** Use the cursor to select the triangle first and then the x-axis to reflect △ABC about the x-axis as shown in figure 2.
- Select **Coord. & Eq.** to display coordinates of the vertices of the image in quadrant IV.

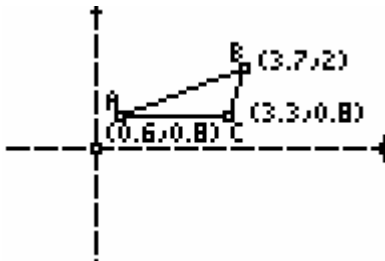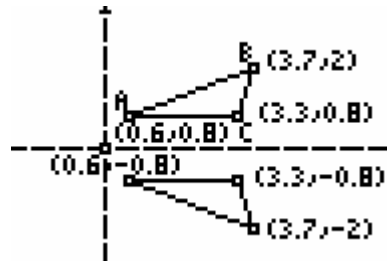Figure 1   Figure 2

Preimage Coordinates		Image Coordinates	
x	y	x'	y'
0.6	0.8	0.6	-0.8
3.7	2	3.7	-2
3.3	0.8	3.3	-0.8

a. The x-coordinates stayed the same and the y-coordinates of the image are the opposite of the corresponding y-coordinates of the preimage. Corresponding side lengths and angle measures can be shown to stay the same using **Measure** on the menu. Area and perimeter can also be shown to stay the same under a reflection.

b. $(x, y)) \rightarrow (x, -y)$

3. **Reflect the sail about the line y=x.**
   - Draw a set of axes on your graphing calculator screen using **Hide/Show-axes**.
   - Draw a triangle in the first quadrant on one side of the line y=x and label it △ABC.
   - Use **Coord. & Eq.** to display the coordinates of the vertices as shown in figure 1.
   - Select **Reflection** and reflect △ABC about y=x as shown in figure 2.
   - Label the coordinates of the vertices of the image (figure 2).
   - Complete the table below and answer the questions that follow.
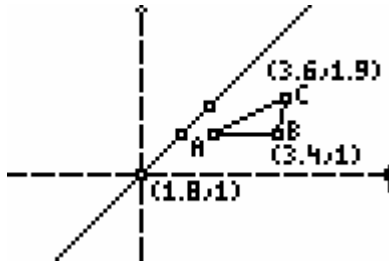
Figure 1		Figure 2
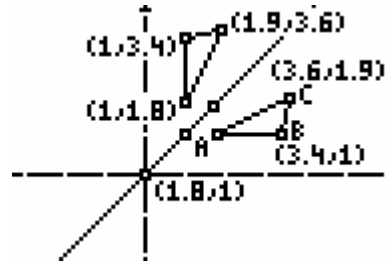
Preimage Coordinates		Image Coordinates	
x	y	x'	y'
1.8	1	1	1.8
3.4	1	1	3.4
3.6	1.9	1.9	3.6

a. The x-coordinates and the y-coordinates of the image are the reverse of the corresponding coordinates of the preimage.

b. **(x, y)→(y, x)**

4. **Sequence of Reflections:**
   Reflect the sail about the x-axis and then the y-axis.
   - Draw a triangle in the first quadrant and label it △ABC as shown in figure 1.
   - Use **Coord. & Eq.** from F5 to display the coordinates of the vertices.
   - Press F4 and select **Reflection**. Select △ABC and then the x-axis to reflect it about the x-axis.
   - Select **Coord. & Eq.** to display the coordinates of the image in quadrant IV (figure 1).
   - Press F4 and select **Reflection** to reflect the triangle in quadrant IV about the y-axis as shown in figure 2.
   - Use **Coord. & Eq.** to display the coordinates of the vertices of the new image.
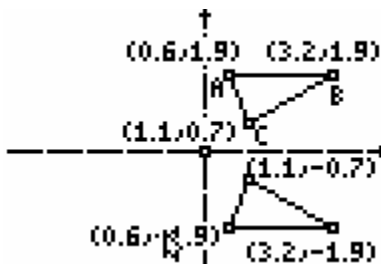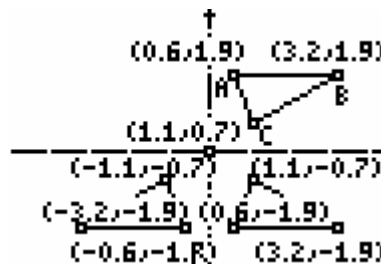   - The data is recorded in the table below.

Figure 1		Figure 2

Preimage Coordinates		Coordinates for reflection about x-axis		Coordinates for reflection about the y-axis	
x	y	x'	y'	x"	y"
0.6	1.9	0.6	-1.9	-.06	-1.9
3.2	1.9	3.2	-1.9	-3.2	-1.9
1.1	0.7	1.1	-0.7	-1.1	-0.7

a. The coordinates of the final image have opposite signs from the corresponding coordinates of the preimage.

b. **(x, y) → (-x, -y)**

Sequence of Reflections:
5. **Reflect the sail about the y-axis and then about the x-axis.**
    - Draw a set of axes on the screen using **Hide/Show-axes**.
    - Use **Triangle** to draw a triangle in quadrant I and label its coordinates (figure 1).
    - Use **Reflection** to reflect this triangle about the y-axis (shown in quadrant II, figure 2).
    - Use **Coord. & Eq.** to display the coordinates of the vertices of the reflected image.
    - Use **Reflection** to reflect the triangle in quadrant II about the x-axis (figure 3).
    - Use **Coord. & Eq.** to display the coordinates of the vertices of the reflected image.
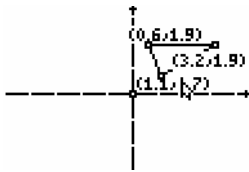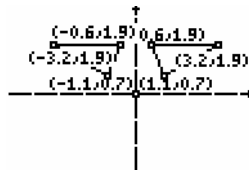   - The data is recorded in the table that follows.
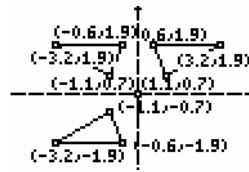

Figure 1   Figure 2   Figure 3

Preimage Coordinates		Coordinates for reflection about y-axis		Coordinates for reflection about the x-axis	
x	y	x'	y'	x"	y"
0.6	1.9	-.06	1.9	-0.6	-1.9
3.2	1.9	-3.2	1.9	-3.2	-1.9
1.1	0.7	-1.1	0.7	-1.1	-0.7

a. The coordinates of the final image are opposite in sign of the corresponding coordinates of the preimage.

b. **(x, y) → (-x, - y)**

c.   Order does not matter. A reflection of a geometric figure about the x-axis then the y-axis results in the same image as a reflection of the same geometric figure about the y-axis then the x-axis. Both sequences result in the same final image. The tables in # 4 and # 5 show that the coordinates in the third column are the same.

Translation Investigations:
6. **Translation in one direction:**
   Translate the sail to the right 4 units.
   - Draw a set of axes on the graphing calculator screen using **Hide/Show-axes.**
   - Select **Triangle** from F2 and draw △ABC in quadrant II as shown in figure 1.
   - Press F5 and select **Coord. & Eq.**. Select each vertex and press ENTER to display the coordinates of the vertices.
   - Press F2 and select **Point-on**. Place point T on the positive x-axis at a distance from the origin (point O) equal to a distance you want to translate △ABC. This distance can be adjusted later to correspond to 4 units.
   - To translate △ABC, press F4 and select **Translation.**
   - To translate △ABC to the right by the distance OT, move the cursor to a side of the triangle and press ENTER. Then move the cursor to point O followed by point T pressing ENTER each time.
   - Check a pair of corresponding coordinates to see if △ABC has been translated 4 units to the right. If not, press CLEAR, select point T, and then press ALPHA. This will allow you to move point T along the x-axis until the coordinates indicate a change of 4 units horizontally to the right as shown in figure 2.
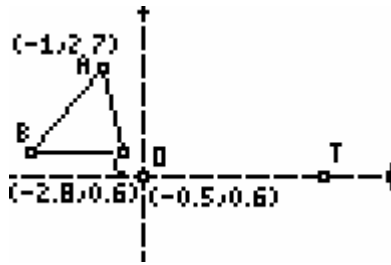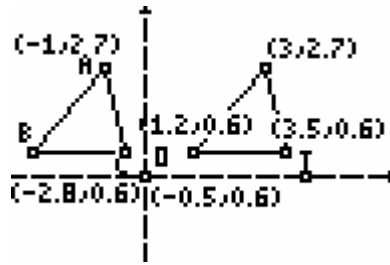   - The data is recorded in the table that follows.

Figure 1

Figure 2

Preimage Coordinates		Image Coordinates	
x	y	x'	y'
-1	2.7	3	2.7
-2.8	0.6	1.2	0.6
-0.5	0.6	3.5	0.6

   a. The x-coordinates of the image are 4 units more than the corresponding x-coordinates of the preimage. The corresponding y-coordinates stayed the same.

   b. Using **Measure** from F5, the corresponding sides and angles can be measured. It can be shown that the corresponding sides and corresponding angles did not change under this transformation. Area and perimeter can also be verified to be the same.

   c. $(x, y) \rightarrow (x + 4, y)$

7. **Translate the sail to the left 5 units.**

- Select **Hide/Show-axes** in F5 and draw a set of axes on the graphing calculator screen.
- Select **Triangle** in F2 and draw a triangle in quadrant I labeling the vertices as shown in figure 1.
- Select **Coord. & Eq.** to display the coordinates of the vertices of △ABC (figure 1).
- Use **Point-on** in F2 to mark point M on the negative x-axis as shown in figure 1.
- Press F4 and select **Translation**. Move the cursor to select △ABC followed by point O and point M to complete the translation of △ABC left by a distance equal to OM.
- Check the coordinates for a change of 5 units left. If an adjustment is needed, press CLEAR. Select point M followed by ALPHA to get a picture of a hand on the screen. This hand can be used to move point M until the desired coordinates are displayed.
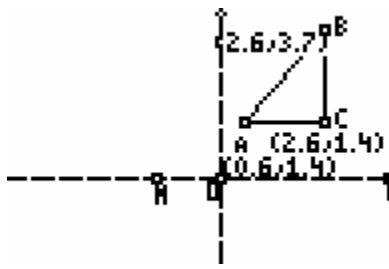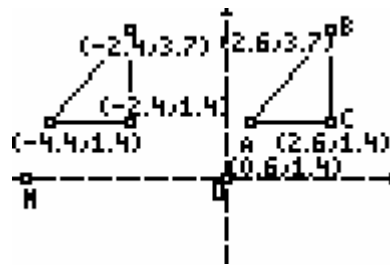- The data is recorded in the table that follows.

Figure 1	Figure 2

Preimage Coordinates		Image Coordinates	
x	y	x'	y'
0.6	1.4	-4.4	1.4
2.6	3.7	-2.4	3.7
2.6	1.4	-2.4	1.4

a. From the table and graph, it can be seen that the corresponding y-coordinates stayed the same. However, the absolute value of the difference of the corresponding x-coordinates is 5. **Measure (D. & Length or angle)** in F5 can be used to show corresponding angles and corresponding sides congruent. Area and perimeter can also be shown to be the same for the image and preimage.

b. **(x, y) → (x – 5, y)**

8. **Translate the sail by a distance OT up in a vertical direction where point O is the origin and point T is any point on the positive y-axis.**
- Draw a set of axes on the calculator screen using **Coord. & Eq.** in F5.
- Draw a triangle in quadrant I near the x-axis, label its vertices, and display the coordinates of its vertices as shown in figure 1.
- Mark a point T on the positive y-axis to represent the distance for the translation vertically as shown in figure 1.
- Select **Translation** in F4.
- Move the cursor on the screen to select △ABC, the origin, and point T in that order. Then press ENTER to view the translation on the screen.
- The translation of △ABC by a distance OT up in a vertical direction can be seen in

figure 2.
- Use **Coord. & Eq.** to display the coordinates of the vertices of the image.
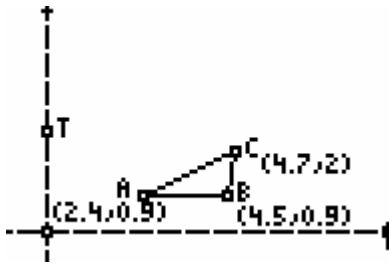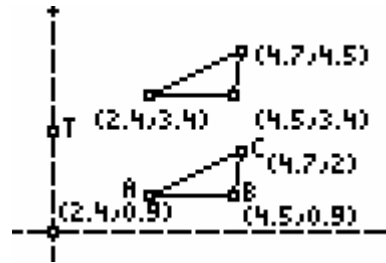- The data is recorded in the table that follows.

Figure 1

Figure 2

Preimage Coordinates		Image Coordinates	
x	y	x'	y'
2.4	0.9	2.4	3.4
4.5	0.9	4.5	3.4
4.7	2	4.7	4.5

a. This vertical translation did not affect the corresponding x-coordinates. However, the absolute value of the difference of the corresponding y-coordinates is 2.5 indicating a vertical translation of 2.5 units.

b. **(x, y) → (x, y + 2.5)**

9. **Translate the sail a distance of OT down where point O is at the origin and point T is a point on the negative y-axis.**

- Draw a set of axes on your graphing calculator screen using **Hide/Show-axes.**
- Press CLEAR and move the cursor to an axis or the origin and press ALPHA. Use the hand to move the axes on the screen so that mostly quadrant I and IV are viewed on the screen.
- Draw a triangle (sail) in quadrant I and mark a point T on the negative y-axis as shown in figure 1. OT represents the distance that the preimage will be translated down.
- Select **Coord. & Eq.** to display the coordinates of the vertices of the preimage.
- Select **Translation** in F4. Then move the cursor to select the preimage, point O, and point T. Press ENTER to view the translation on the screen.
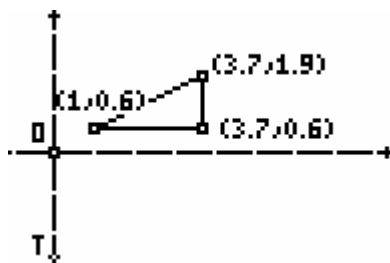- The coordinates of the preimage and image are recorded in the table below.
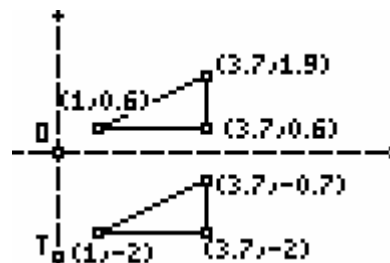

Figure 1

Figure 2

Preimage Coordinates		Image Coordinates	
x	y	x'	y'
1	0.6	1	-2
3.7	0.6	3.7	-2
3.7	1.9	3.7	-0.7

a. The corresponding x-coordinates stayed the same; each y-coordinate of the image is obtained by subtracting 2.6 from the corresponding y-coordinate of the preimage.

b. $(x, y) \rightarrow (x, y - 2.6)$

c. $(x, y) \rightarrow (x, y +/- b)$

10. **Sequence of translations:**
Translate the sail OM to the left and OP down where point O is at the origin, point M is a point on the negative x-axis, and point P is on the negative y-axis.

A sequence of translations is shown below in figures 1, 2, and 3. Figure 1 shows ΔABC (the sail) in its starting position. In figure 2, the sail has been translated a distance OM to the left. Figure 3 shows this image translated a distance OP down. A sequence of translations is shown in figure 3.
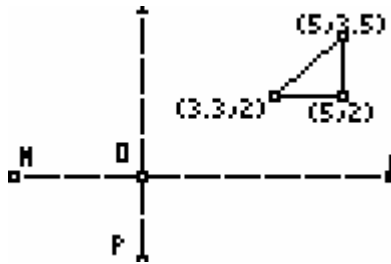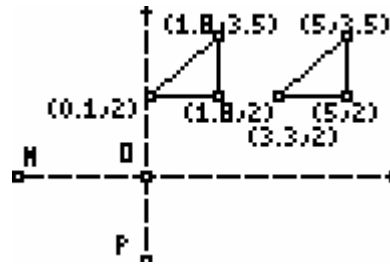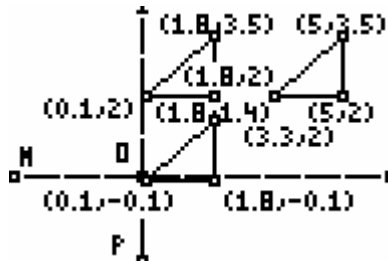

Figure 1

Figure 2

Figure 3

The coordinates of the original sail and its final image are recorded in the table below. Study the graph and table to answer the questions that follow.

Preimage Coordinates		Image Coordinates	
x	y	x'	y'
3.3	2	0.1	-0.1
5	2	1.8	-0.1
5	3.5	1.8	1.4

a. The x-coordinates of the image are 3.2 units less than the corresponding x-coordinates of the original sail (preimage). The y-coordinates of the image are 2.1 units less than the corresponding y-coordinates of the original sail.

b. **(x, y) → (x -3.2 , y – 2.1)**

11. Translate the sail OP down and then OM left.
Start with the sail in the same position as figure 1 in #9 above (shown in figure 1 below). Translate the sail (triangle) a distance of OP down as shown in figure 2. Then translate the sail in its second position a distance of OM to the left as shown in figure 3. The coordinates of the original triangle (sail) and corresponding coordinates of its final position are recorded in the table below. The graph can be used to answer the questions on the Activity Sheet *Set the Sails*.
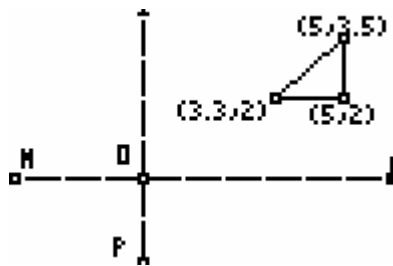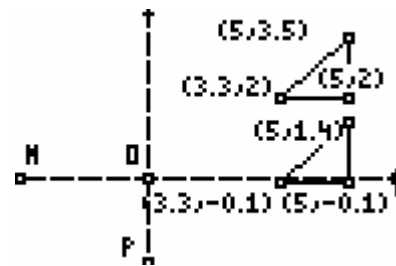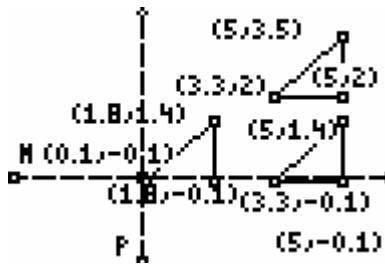

Figure 1

Figure 2

Figure 3

a. The sequence of translations in this problem resulted in a triangle in the same position as the final position of the triangle in problem #9.

b. A comparison of the coordinates of corresponding vertices for the final image in #9 and #10 indicates that order did not matter in this sequence of translations.

12. Sequence of Reflections and Translations:
Use a TI-83+ with Cabri Junior™ software to do the following investigations. Start with a sail (triangle) in a coordinate plane on your calculator screen.
 a. Reflect the sail about the y-axis and then translate the image down a distance of OP.
 b. Translate the sail down by a distance of OP and then reflect it about the y-axis.
 c. Compare the results and determine if this sequence of transformations produces the same result. Does order matter in this sequence of transformations? Why or why not?

13. Rotations in the coordinate plane:
   • Select **Hide/show-axes** in F5 and press ENTER to draw a set of axes in a coordinate plane.
   • Press F2 and select **Triangle**. Draw a triangle (sail) in quadrant II as shown in figure 1.
   • Press F5 and select **Coord. & Eq.**. Move the cursor to each vertex and press ENTER each time to display the coordinates of the vertices of the triangle. Use the "hand" to move the coordinates as needed.
   • Press F2 and select **Circle**. Locate a point in the middle of quadrant III and press ENTER to mark the center of the circle. Then move the cursor away from the center point to a radius that keeps the circle in quadrant III.
   • Use **Point-on** to locate another point on the circle.
   • Press F2 and select **Segment.** Draw radii as shown in figure 1 using this option.
   • Press F5 and select **Measure-angle**. Measure the angle formed by the two radii.
   • Press CLEAR and move the cursor to an endpoint of one radius and press ALPHA.
   • Move the "hand" around the circle until the degree measure of the angle shows 90.
   • Press F4 and select **Rotation**. There are 5 steps that need to be followed in order to complete the rotation. Use the cursor to select:
           a) the triangle (1 step).
           b) the center of rotation (origin in this case). (1 step)
           c) the angle of rotation (3 steps- point on side, vertex, point on other side)

   • The 90 degree rotation of the original triangle about the origin is shown in figure 2.
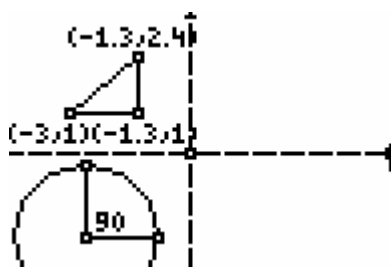   • Figure 3 shows the original triangle rotated 180 degrees about the origin.
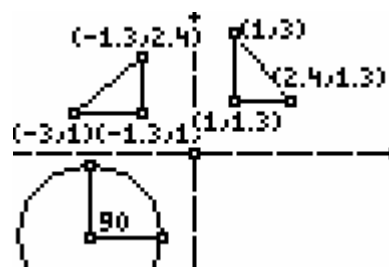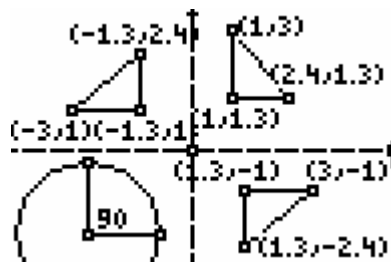

Figure 1

Figure 2

Figure 3

The table below shows the coordinates of the triangle (sail) in its original position, rotated 90 degrees clockwise about the origin, and rotated 180 degrees clockwise about the origin.

Coordinates of Original Sail (Triangle)		Coordinates of sail rotated 90 degrees clockwise		Coordinates of sail rotated 180 degrees clockwise	
x	**y**	**X'**	**Y'**	**X"**	**Y"**
-3	1	2.4	1.3	3	-1
-1.3	1	1	3	1.3	-1
-1.3	2.4	1	1.3	1.3	-2.4

a. When the sail (triangle) is rotated 180 degrees clockwise about the origin, the corresponding x-and y-coordinates have the same magnitude and opposite signs.

b. **(x , y) → (- x, - y)**

Extension:
Investigate the rotation of the original sail in #11 on the Activity Sheet *Set the Sails!* for a clockwise rotation of 270 degrees and compare the resulting coordinates with those of the original.

Solution: **(x, y) → (-y, x)**

References:
T³• Texas Teachers Teaching with Technology (2003). *Cabri Junior*[TM] *(*Participant Packet). Texas Instruments Incorporated. Dallas, TX.
www.education.ti.com
www.education.ti.com/support