Ratio and Proportion

Ratio: The concept of "ratio" occurs frequently and in a wide variety of ways. For example:

- A newspaper reports that the ratio of Republicans to Democrats on a certain Congressional committee is 3 to 2.
- The student/faculty ratio at ABC University is 30 to 1.
- The speed limit on a certain interstate highway is 70 miles per hour. (In English the word "per" means "for each" and indicates a ratio; in this case 70 miles to each hour.)
- A grocery store advertises eggs @ $1.10 per dozen. This is the ratio 1.10 to 1.
- If the probability that a certain event E will occur is a, then the probability that E will not occur is $1 - a$. The odds that E will occur is the ratio a to $1 - a$, the odds that E will not occur is the ratio $1 - a$ to a, where $0 < a < 1$.
- In geometry, "similarity of triangles" is expressed in terms of ratios of corresponding sides.
- A map represents a portion of the earth's surface. The map's scale is stated as a proportion, e.g., 1 cm = 25 km is the ratio 1 : 2,500,000.

A *ratio* is a comparison of two numbers a and b with $b \neq 0$. Ratios are expressed verbally as "a to b", by the symbol $a : b$, or as the fraction $\frac{a}{b}$.

Example: In a certain college English class there are 21 students, 13 females and 8 males.

a. The ratio of females to males is 13 to 8, or $\frac{13}{8}$.

b. The ratio of males to females is 8 to 13, or $\frac{8}{13}$.

c. The ratio males to students in the class is 8 to 21, or $\frac{8}{21}$.

d. The ratio of students to females is 21 to 13, or $\frac{21}{13}$.

The example illustrates that we can form ratios in a variety ways. If we think of the 21 students as the whole, then the ratios in (a) and (b) are *part to part* comparisons, (c) gives a *part to whole* comparison, and (d) gives a *whole to part* comparison.

Ratios represent relative amounts as opposed to absolute amounts. Knowing that the ratio of Republicans to Democrats on a certain committee is 3 to 2 does not tell us how many Republicans and how many Democrats are actually on the committee. There could be 3 Republicans and 2 Democrats on a committee of 5 members, or 9 Republicans and 6 Democrats on a committee of 15 members, and so on. If the directions for making a special color of paint state that 5 parts of blue paint to 3 parts of white paint are required, then the amount of blue and white paint we mix will depend on how much of the special paint we need. For example, if we need 4 gallons of the special paint, then we will mix 5/2 gallons (= 10 quarts) of blue with 3/2 gallons (= 6 quarts) of white.

Here are some more examples.

Examples:

1. The ratio of student/faculty ratio at ABC University is 30 to 1.
   a. If there are 12,000 students enrolled at the university, how many faulty members are there?
   b. Suppose the university has 600 faculty members. How many students are enrolled?

2. Suppose that the ratio of females to males in the English class above represents the ratio of females to males for the entire college.

   a. If there are 800 males at the college, how many females are there?
   b. If there are 6,300 students enrolled at the college, how many are females and how many are males?

Solutions:

1.

a. There are 30 students for each faculty member. That is, there are 30 times as many students as there are faculty members. If we let x = the number of faculty members, then there are 30x students. Solving the equation

$$30x = 12{,}000 \quad \text{gives} \quad x = 400.$$

Thus, there are 400 faculty members at the university. Another way to arrive at this result is to write the ratios as equal fractions: $\frac{\text{students}}{\text{faculty}}$.

$$\frac{30}{1} = \frac{12{,}000}{x}$$

and solve for x. Cross-multiplying gives $30x = 12{,}000$ and $x = 400$.

b. If the university has 600 faculty members, then there must be 30 × 600 = 18,000 students.

2. a. If we let x denote the number of females at the college, then we must have

$$\frac{13}{8} = \frac{x}{800} \quad \text{which implies} \quad x = 1300.$$

b. The ratio of females to students in the class is 13 to 21 and this same ratio applies to the whole college. If we let x = the number of female students in the college, then we have

$$\frac{13}{21} = \frac{x}{6{,}300}$$
$$21x = 13(6{,}300)$$
$$x = 3{,}900$$

There are 3,900 females and 6,300 – 3,900 = 2,400 males enrolled at the college.

Proportion: The solutions in our example above illustrate the idea of equal ratios. Two ratios are said to be *proportional* if and only if the fractions that represent the ratios are equal. Two equal ratios form a *proportion.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two ratios. Then, by our definition, these two ratios are proportional if and only if

$$\frac{a}{b} = \frac{c}{d} \quad \text{which is equivalent to} \quad ad = bc$$

after cross-multiplying.

Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are proportional. Since b and d are nonzero, there is a number r such that $d = rb$ and

$$\frac{a}{b} = \frac{c}{d} = \frac{c}{rb} \quad \text{that is} \quad \frac{a}{b} = \frac{c}{rb}.$$

Cross-multiplying, we get

$$rab = bc \quad \text{which implies} \quad c = ra \quad \text{after dividing both sides by } b.$$

This gives us another characterization of proportions: $\frac{a}{b} = \frac{c}{d}$ if and only if $c = ra$ and $d = rb$ for some nonzero number r.

If a and c are nonzero, then

$$\frac{a}{b}=\frac{c}{d} \quad \text{is equivalent to} \quad \frac{b}{a}=\frac{d}{c}.$$

That is, $\frac{a}{b}$ is proportional to $\frac{c}{d}$ if and only if $\frac{b}{a}$ is proportional to $\frac{d}{c}$, when $a, b, c,$ and d are all nonzero.

A variety of problems can be solved using proportions. Here are some examples.

Examples:

1. Your car averages 29 miles per gallon of gas. How many gallons of gas will you need for a 609-mile trip?

2. In an architect's drawing 0.5 centimeters represents 9 meters.
   a. How many meters will 4 centimeters represent?
   b. How many centimeters will represent 36 meters?

3. A recipe that will serve 6 people requires 4 eggs. How many eggs will be needed if 15 people are to be served?

4. A rectangular yard has a width to length ratio of 5 : 9. If the perimeter of the yard is 2800 feet, what are the dimensions of the yard?

Solutions:

1. We'll display the information in a table:

	Average	Trip
Miles	29	609
Gallons	1	x

From the table, we get

$$\frac{29}{1}=\frac{609}{x}.$$

Cross-multiplying and solving for x, we get

$$29x = 609 \quad \text{and} \quad x = 21.$$

Therefore, you will need 21 gallons of gas for the 609-mile trip.

2. a. Let x = the number of meters. The proportion is

$$\frac{0.5}{9} = \frac{4}{x}$$

Cross-multiplying and solving for x gives

$$0.5x = 39 \quad \text{which implies} \quad x = 72.$$

Four centimeters will represent 72 meters.

b. Let x = the number of centimeters. This time the proportion is

$$\frac{0.5}{9} = \frac{x}{36}$$
$$9x = 18$$
$$x = 2.$$

Two centimeters will represent 36 meters.

3. Let x = the number of eggs. The proportion is

$$\frac{4}{6} = \frac{x}{15}.$$

Cross-multiplying, we get

$$6x = 60 \quad \text{and} \quad x = 10.$$

Ten eggs will be needed.

4. Let x = the width and let y = the length. Since the perimeter of the rectangle is 2800 feet, we have

$$2x + 2y = 2800 \quad \text{which implies} \quad x + y = 1400 \quad \text{and} \quad y = 1400 - x.$$

Since the ratio of the width to length is $5 : 9$, we have

$$\frac{5}{9} = \frac{x}{1400 - x} \quad \text{which implies} \quad 5(1400 - x) = 9x.$$

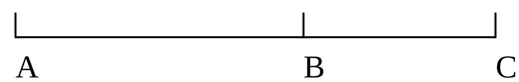
Solving this equation for x, we have

$$7000 - 5x = 9x$$
$$14x = 7000$$
$$x = 500.$$

The dimensions of the yard are: width 500 ft, length: 900 ft.

The ancient Greeks said that a line segment $\overline{AC}$ is divided into the *golden ratio* by the point B if $\overline{AB}/\overline{BC}$ is proportional to $\overline{BC}/\overline{AC}$; that is if

$$\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{BC}}{\overline{AC}}.$$



If we let $\overline{AB} = x$ and $\overline{BC} = 1$, then we have

$$\frac{x}{1} = \frac{1}{1+x} \quad \text{and} \quad x^2 + x = 1.$$

This yields the quadratic equation $x^2 + x - 1 = 0$ whose roots are:

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $x \geq 0$, it follows that $x = \dfrac{\sqrt{5}-1}{2} \approx 0.618$. This is the golden ratio.

Psychological tests have shown that to most people the most “pleasing” rectangle (in terms of proportion) is the rectangle whose ratio of width to length is the golden ratio. Such a rectangle is called a *golden rectangle.* The golden rectangle was used by the Greeks in their architecture and their art.

The golden ratio is closely related to the Fibonacci sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots\ .$$

The ratio of successive terms, a_k / a_{k+1}, of this sequence approaches the golden ratio as a limit; that is,

$$\frac{a_k}{a_{k+1}} \to \frac{\sqrt{5}-1}{2} \quad \text{as} \quad k \to \infty.$$

Scaling: Take a rectangle with width w and height h. Then the perimeter P and the area A of the rectangle are given by

$$P = 2w + 2h \quad \text{and} \quad A = wh.$$

Suppose that we double the width and double the length. Then the dimensions of the new rectangle are $2w$ and $2h$. The perimeter P' and area A' are

$$P' = 2(2w) + 2(2h) = 2P \quad \text{and} \quad A' = (2w)(2h) = 4wh = 2^2 wh.$$

We've doubled the perimeter and quadrupled the area. Equivalently, the ratios of the new perimeter to the old and the new area to the old are:

$$\frac{P'}{P} = 2 \quad \text{and} \quad \frac{A'}{A} = 2^2.$$

In general, if we change the width and height of a rectangle by a factor c, then the new perimeter is c times the old perimeter, and the new area is c^2 times the old area.

$$P' = 2(cw) + 2(ch) = c[2w + 2h] = cP \quad \text{and} \quad A' = (cw)(ch) = c^2 wh = c^2 A.$$

The ratios of the new perimeter to the old and the new area to the old are

$$\frac{P'}{P} = c \quad \text{and} \quad \frac{A'}{A} = c^2.$$

In fact, as you can verify by looking at the formulas, this result holds for any plane figure (triangle, pentagon, hexagon, . . . , circle).

In the same way, if the dimensions of a solid figure (e.g., a prism, pyramid, cylinder, cone, . . .) are changed by a factor c, then the surface area is changed by the factor c^2 and the volume is changed by the factor c^3. Equivalently, the ratios of the new surface area S' to the old surface area S, and the new volume V' to the old volume V are:

$$\frac{S'}{S} = c^2 \quad \text{and} \quad \frac{V'}{V} = c^3.$$

Example: A right circular cone of radius r and height h has (total) surface area

$$S = \pi r^2 + \pi r l$$

where r is the radius and l is the slant height, and volume

$$V = \frac{1}{3}\pi r^2 h$$

where r is the radius and h is the height.

If we change each of the dimensions by the factor c, then the new cone has radius cr, slant height cl, and height ch. The total surface area of the new cone is

$$S' = \pi(cr)^2 + \pi(cr)(cl) = c^2[\pi r^2 + \pi rl] = c^2 S \ ,$$

and the volume of the new cone is

$$V' = \frac{1}{3}\pi(cr)^2(cl) = c^3\left[\frac{1}{3}\pi r^2 h\right] = c^3 V.$$