MEASUREMENT

Introduction: People created systems of measurement to address practical problems such as finding the distance between two places, finding the length, width or height of a building, finding the area of a plot of land, finding the capacity of a container, and finding the weight of food products. They constructed calendars to measure the passage of time.

Every measurement requires a unit of measure. In ancient times, people used body measurements as units of length, seeds and stones as units of weight, and the cycles of the sun and moon as units of time. Today there are two systems of measurement in common use, the metric system and the customary system, also called the British system or the old English system. Since England and the United States use slightly different liquid measures, the customary system is sometimes called the American/British system

Most countries in the world use the metric system. Australia and the United States are the exceptions; both use the customary system. Beginning with the "space age" in the early 1960's there have been attempts to convert the United States to the metric system but these have not been successful. However, there are areas where the metric system is used widely in the United States, most notably in the sciences.

Nonstandard Units of Measurements.

Historical records indicate that the first units of length were based on people's hands, feet and arms. The measurements were:

1. One hand – the width of an adult's hand. Today this unit of measurement is taken to be 4 inches and it is used to measure the height of horses.
2. One foot – the length of an adult's foot.
3. One cubit – the length of an adult's forearm, measured from the elbow to fingertip.

While these units were readily available, there were obvious disadvantages – different people have different-sized hands, feet and arms. The lack of uniformity in these measurements probably resulted in choosing one individual (e.g. a national ruler) and specifying the width of that person's hand, or the length of the foot or arm as the "standard" unit of length. This "unit" length could then be transferred to a surface and reproduced.

Standard Units of Measurement.

The customary (British) system: The *customary system* was actually developed from nonstandard units. One inch was the length of three barleycorns placed end to end, the foot was the length of an adult's foot, and the yard was the distance from the tip of a person's nose to the end of an outstretched arm. The "foot" was standardized in the late 18^{th} century and the inch, the yard and the mile were defined in terms of a foot.

The metric system: The *metric system* was developed in France in the 18th century. The word "metric" comes from the Latin word "metricus" or from the Greek word "metron," both meaning measure. The metric system is based on the meter. Originally the meter was intended to be 1/10,000,000 of the distance measured on the earth's surface from the equator to the North Pole. However, in spite of careful surveying, the accuracy of that computation was sufficiently doubtful to cause the legal definition of the meter to be changed to the length of a certain platinum bar kept in Paris. Because the length of a metal bar is subject to variations in its environment (e.g., the temperature), the meter today is defined to be the distance light travels in a vacuum in $\frac{1}{299{,}792{,}458}$ seconds.

The advantages of the metric system over the customary system are: (1) all conversions within the system are based on powers of 10; it is a decimal system; and (2) the metric prefixes are the same for length, liquid volume and weight. The prefixes are:

Prefix	kilo-	hecto-	deka-		deci-	centi-	milli-
Meaning	1000	100	10	1	1/10	1/100	1/1000

Basic units and conversion factors: The basic units for the measurement of length and the measurement of liquids in the customary and metric systems, and the conversions between the two systems are given in the following tables.

Linear Measure

Customary System	**Metric System**	**Conversion Table**
12 inches (in) = 1 foot (ft)	10 millimeters (mm)=1 centimeter (cm)	1 cm = 2.54 in
3 feet = 1 yard (yd)	10 centimeters = 1 decimeter	0.33937 in = 1cm
5 ½ yards = 1 rod	10 decimeters = 1 meter (m)	1 ft = 0.3048 m
5, 280 feet = 1 mile (mi)	10 meters = 1 dekameter	1 m = 3.28 ft
	1,000 meters = 1 kilometer (km)	1 mi = 1.609 km
		0.621 mi = 1 km

Liquid Measure

Customary System	**Metric System**	**Conversion Table**
16 ounces (oz) = 1 pint (pt)	10 milliliters (ml) = 1 centiliter (cl)	1 oz = 2.9573 cl
2 cups = 1 pint	100 centiliters = 1 liter (l)	1 cl = 0.3382 oz
2 pints = 1 quart (qt)	1,000 liters = 1 kiloliter (kl)	1 qt = 0.9463 l
4 quarts = 1 gallon (gal)		1 l = 1.0567 qt
		1 gal = 0.0038 kl
		1 kl = 264.179 gal

Weight vs Mass: People confuse mass and weight. *Mass* measure the quantity of matter that a body contains. *Weight* measures the force of gravity on a body. As a simple illustration of the difference, a man who weighs 180 pounds here on earth has a mass of 82 kilograms. The force of gravity on the moon is 1/6 the force of gravity on the earth. Therefore, on the moon, the man would weigh 30 pounds, but his mass is still 82 kilograms. The relationship between weight (w) and mass (m) is:

$$m = \frac{w}{g}$$

where g is the acceleration due to gravity; g is approximately 32 ft/sec^2 on the earth. Although we will not use it here, the basic unit of mass in the customary system is the *slug*; 1 slug = 32 pounds.

The basic units for the measurement weight/mass in the customary and metric systems, and the conversions between the two systems are given in the following table.

Weight/Mass

Customary System	**Metric System**	**Conversion Table**
16 ounces (oz) = 1 pound (lb)	10 milligrams (mg) = 1 centigram (cm)	1 oz = 28.3495 g
2000 pounds (lbs) = 1 ton	10 centigrams = 1 decigram (dg)	1 g = 0.0353 oz
	10 decigrams = 1 gram (g)	1 lb = 0.4536 kg
	1000 grams = 1 kilogram (kg)	1 kg = 2.2046 lbs
	1000 kilograms = 1 metric ton	1 ton = 0.9072 metric tons
		1 metric ton = 1.1023 tons

Time: The basic unit of time is the *second*. This is the unit used independent of the choice of the customary or metric system for measuring length, liquids or weight.

The solar "clock" was originally used to define the second. A "solar day" is the interval of time that elapses between two successive crossings of the same meridian by the Sun at its highest point in the sky at that meridian. By this measure, a second was defined to be 1/86,400 of a solar day. However, variations in the elliptical path of the earth around the sun cause solar days to vary in length. In 1967, an atomic standard was adopted as a more precise unit of measure. The atomic "clock" keeps time with an accuracy of about 3 millionths of a second per year.

Precision, Significant Figures, Error

Precision: In discussing measurement we need to distinguish between two types of numbers. *Exact numbers* are numbers without any uncertainty or error. The number 2 in the equation $r = d/2$ which relates the radius and diameter of a circle, and the number 0.06 in the equation $A(t) = P(1 + 0.06t)$ which gives the amount of money in an account earning simple interest at 6% on a principal P, are examples of exact numbers. In contrast, *measured numbers,* or *measured quantities,* are numbers arising from some measurement process. Measured numbers by their very nature involve some degree of uncertainty or error. For example, if a man states that his height is 5 feet 11 inches, he does not mean that his height is *exactly* 5 feet 11 inches (although, by chance, it may be), he means that within the accuracy of the measuring device (say a tape measure) his height is 5 feet 11 inches to the nearest inch.

Example: Measure the length of an 8 ½ by 11 sheet of paper to:

a. The nearest centimeter.
b. The nearest millimeter.

Our results are: to the nearest centimeter: 28 cm; to the nearest millimeter: 279.5 mm.

The *precision* of a measurement is determined by the smallest unit of measurement used. To the nearest foot, the man's height is 6 feet; to the nearest inch, his height is 5 feet 11 inches. The latter is a more precise measurement. In the case of the metric length of an 8 ½ by 11 sheet of paper, 279.5 mm is a more precise measurement than 28 cm.

Significant figures: The degree of accuracy of a measured quantity depends on the measuring scale of the instrument used to make the measurement. The more finely divided the scale, the more accurate the measurement. For example, using the centimeter scale on our ruler, we found that the length of our 8 ½ by 11 sheet of paper was 28 centimeters; using the millimeter scale, we found that the length was 279.5 millimeters. The millimeter scale provides more significant figures and a greater degree of accuracy.

The number of *significant figures* (also called *significant digits*) of a measured quantity is the number of reliably known digits it contains. In our sheet of paper example, 28 cm has two significant figures and 279.5 mm has four significant figures. Confusion sometimes arises when a quantity contains one or more zeros. For example, how many significant figures does 0.0376 cm have? What about 203.5 m or 5725.0 m? The conventions concerning significant figures and zeros are as follows:

1. Zeros at the beginning of a number are not significant; they merely locate the decimal point.

   0.0376 cm has three significant figures (3, 7, 6)

2. Zeros within a number are significant.

   203.5 m has four significant figures (2, 0, 3, 5)

3. Zeros at the end of a number after the decimal point are significant.

5725.0 m has five significant figures (5, 7, 2, 5, 0)

4. Zeros at the end of a whole number are *not* significant.

5500 kg has two significant figures (5, 5)

Unfortunately, there is no general consensus on #4. Using the number 5500 kg, for example, we don't know whether the measurement was made to the nearest hundred feet, to the nearest ten feet, or even to the nearest foot. In contrast to the convention above, some authors take the opposite view and state that zeros at the end of a whole number are significant.

It is important to express the results of mathematical operations with the proper number of significant figures. If a mathematical operation involves multiplication or division then:

The final result should have the same number of significant figures as the quantity with the least number of significant figures that was used in the calculation.

Rounding: The rules for *rounding off* numbers are as follows:

1. If the next digit after the last significant figure is 5 or greater, the last significant figure is increased by 1.
2. If the next digit after the last significant figure is less than 5, the last significant figure is left unchanged.

For multiple operations involving multiplication and/or division, rounding off to the proper number of significant figures should not be done at each step since rounding errors may accumulate. Rounding off should be done on the final result.

Examples:

1. Multiplication:

$$\underset{(2\text{ sf})}{3.6\text{ m}} \times \underset{(3\text{ sf})}{2.71\text{ m}} \ (= 9.756\text{ m}^2) = 9.8\text{ m}^2 \quad \text{(rounded to 2 sf)}$$

2. Division:

$$\frac{\overset{(4\text{ sf})}{456.8\text{ ft}}}{\underset{(3\text{ sf})}{1.25\text{ sec}}} (= 365.44\text{ ft/sec}) = 365\text{ ft/sec} \quad \text{(rounded to 3 sf)}$$

If a mathematical operation involves addition or subtraction, then:

> *The final result should have the same number of decimal places as the quantity with the least number of decimal places that was used in the calculation.*

In contrast to multiplication and division, this rule should be applied by rounding off the numbers to the least number of decimal places *before* adding or subtracting.

Example:

32.2		32.2
1.723	→	1.7
12.57	(rounding off)	12.6
		46.5

Absolute error: In making a measurement it is understood that the result is reported as "accurate" to the nearest unit being used in the measurement. For example, if a man says that he weighs 182 lbs that implies that his weight is 182 lbs to the nearest pound. We can conclude, therefore, that his weight is greater than or equal to 181.5 and is less than 182.5; that is, his weight is a number in the interval $[181.5, 182.5)$. The greatest possible error, or absolute error, in measuring the man's weight is 0.5 lbs.

In general, in making a measurement, the *absolute error* is half of the smallest unit used in the measurement.

Examples:

1. A man's shoe size is measured to be 29.3 cm. The smallest unit of measurement is one-tenth of a centimeter. His actual shoe length is between 29.25 cm and 29.35 cm (including 29.25 cm). The absolute error is 0.05 cm.

2. A skyscraper is reported to be 980 feet tall. The smallest unit of measurement is 10 feet; by our convention 980 has two significant figures. The actual height of the building is between 975 ft and 985 ft (including 975 ft). The absolute error is 5 ft.

Relative error: In the examples above, the possible error in measuring the man's shoe was 0.05 cm; the possible error in measuring the height of the building is 5 feet. Which of these is more significant? The relative error of a measurement addresses this question.

The *relative error* of a measurement is the absolute error divided by the measurement.

Examples: Continuing with the examples above,

1. Relative error $= \frac{\text{absolute error}}{\text{measurement}} = \frac{0.05}{29.3} = 0.0017$ or 0.17%.

2. Relative error $\frac{5}{980} = 0.0051$ or 0.51%.

3. If the building's height was reported to be 982 feet tall, then the smallest unit of measure is 1 foot, the building's height is between 981.5 and 982.5 feet tall, and the absolute error 0.5 feet. In this case, we have

   Relative error $= \frac{0.5}{982} = 0.0005$ or 0.05%.

Dimensional Analysis and Unit Analysis.

The basic quantities used in physical descriptions are called *dimensions.* For example, time, length, and mass are dimensions. If you measure the length of one side of a building, you could express the result in feet, or in meters, or in yards. Regardless of which unit of measure your use, the quantity has the dimension of "length."

We'll express dimensional quantities by symbols such as [T] for time, [L] for length, [M] for mass, and [C] for liquid measure. Derived quantities are a combination of dimensions. Some standard examples are:

Area has dimensions: $[L] \times [L] = [L^2]$

The units of square measure in the customary and metric systems are:

Customary System	*Metric system*
144 square inches = 1 square ft	100 square millimeters = 1 square cm
9 square feet = 1 square yd	100 square centimeters = 1 square dm
160 square rods = 1 acre	100 square decimeter = 1 square m
4840 square yards = 1 acre	10,000 square centimeters = 1 square m
640 acres = 1 square mi	

Volume has dimensions: $[L] \times [L] \times [L] = [L^3]$

Some units of cubic measure in the customary and metric systems are:

Customary System	*Metric System*
1728 cubic inches = 1 cubic ft	1000 cubic millimeters = 1 cubic cm
27 cubic feet = 1 cubic yd	1000 cubic centimeters = 1 cubic dm
	1000 cubic decimeters = 1 cubic m

Since distance = velocity × time,

velocity has dimensions: $\frac{[L]}{[T]}$.

Since velocity = acceleration × time,

acceleration has dimensions: $\frac{[L]/[T]}{[T]} = \frac{[L]}{[T^2]}$

It is important to understand that addition and subtraction of quantities can *only* be done when the quantities have the same dimensions.

Examples:

1. 10 m + 20 m = 30 m which in terms of dimensions is [L] + [L] = [L] is correct.

2. 20 sec – 10 sec = 10 sec which in terms of dimensions is [S] – [S] = [S] is correct.

3. 20 ft + 10 qts which in terms of dimensions is [L] + [C]. This makes no sense.

Dimensional analysis: *Dimensional analysis* is a procedure for checking the dimensional consistency of an equation. Recall that an equation is a mathematical equality. Since physical quantities in an equation have both a numerical value and a dimension, the two sides of the equation must be equal not only in numerical value but also in dimension. For this purpose, dimensions can be treated as algebraic quantities. That is, they can be added or subtracted as indicated above, and they can be multiplied and divided, as we did in the case of area, volume, velocity and acceleration.

Examples:

1. 10 ft × 12 ft = 120 ft^2

and

[L] × [L] = $[L^2]$

2. 20 mi/hr × 2 hr = 40 mi

and

$\frac{[L]}{[T]} \times [T] = [L].$

Both sides of these equations are equal, numerically and dimensionally.

A principal use of dimensional analysis is checking whether an equation that has been derived or is being used to solve a problem has the correct form; that is, has the correct dimensions.

Examples:

1. Is the equation $V = \frac{\pi d^2}{6}$, where V is the volume and d is the diameter of a sphere, dimensionally correct?

2. Is the equation $x = x_0 + v_0 t$, where x and x_0 are lengths, v_0 is velocity, and t is time, dimensionally correct?

Solutions:

1. $[L^3] \neq [L^2]$; the equation is not correct; the dimensions are not equal. The correct formula is $V = \frac{\pi d^3}{6}$.

2. $[L] = [L] + \frac{[L]}{[T]} \times [T] = [L] + [L] = [L]$; the equation is dimensionally correct.

Unit analysis: In checking an equation for dimensions it is often more convenient to use the actual units rather than the symbols [T], [L], etc. This variation of dimensional analysis is called *unit analysis.* The following examples illustrate the unit analysis approach.

Examples:

1. Suppose that x and x_0 are measured in meters, v_0 is measured in meters per second, and t is measured in seconds. Use unit analysis to show that the equation

$$x = x_0 + v_0 t$$

is dimensionally correct.

2. Is the equation $x = \frac{v}{2a}$ where x is measured in yards, v in feet/sec, and a in ft/sec^2 dimensionally correct?

Solutions:

1. The equation is $x = x_0 + v_0 t$. Inserting units for the physical quantities, we get

$$m = m + \left(\frac{m}{s}\right)s \quad \text{or} \quad m = m + m \quad \text{(dimensionally correct)}$$

2. The equation is $x = \dfrac{v}{2a}$. If we insert units for the physical quantities, we get

$$m = \frac{m/s}{m/s^2} = \frac{m}{s} \times \frac{s^2}{m} = s \quad \text{or} \quad m = s.$$

The equation is not correct. We could have arrived at the same conclusion using dimensional analysis:

$$[\mathrm{L}] = \frac{[\mathrm{L}]/[\mathrm{T}]}{[\mathrm{L}]/[\mathrm{T}^2]} = \frac{[\mathrm{L}]}{[\mathrm{T}]} \times \frac{[\mathrm{T}^2]}{[\mathrm{L}]} = [\mathrm{T}]$$

The next example illustrates an advantage of unit analysis over dimensional analysis.

Example: Given the equation $x = x_0 + v_0 t + at^2$ where x and x_0 are measured in feet, v_0 is measured in meters per sec, a is measured in meters per sec^2, and t is measured in seconds.

1. Use dimensional analysis to check the equation.

2. Use unit analysis to check the equation.

Solution:

1. In terms of dimensions, the equation is:

$$[\mathrm{L}] = [\mathrm{L}] + \frac{[\mathrm{L}]}{[\mathrm{T}]} \times [\mathrm{T}] + \frac{[\mathrm{L}]}{[\mathrm{T}^2]} \times [\mathrm{T}^2] = [\mathrm{L}] + [\mathrm{L}] + [\mathrm{L}].$$

The equation is dimensionally correct.

2. Inserting units for the physical quantities, we get

$$\mathrm{ft} = \mathrm{ft} + \frac{\mathrm{m}}{\mathrm{s}} \times s + \frac{\mathrm{m}}{\mathrm{s}^2} \times s^2 = \mathrm{ft} + \mathrm{m} + \mathrm{m}\,.$$

While dimensionally correct, the "equation" doesn't make sense because the terms on the right cannot be added without first changing the one of the units of measurement to the other – feet to meters or meters to feet.

Unit analysis also provides a convenient way to convert between units of measurement.

Examples:

1. Convert 450 centimeters to meters.

2. Convert 3 gallons to quarts.

3. Convert 5 feet to centimeters

3. Convert 1500 meters to miles.

Solutions:

We use the conversion tables given above.

1. From the table, 1 m = 100 cm. Therefore, $\frac{1\,\text{m}}{100\,\text{cm}} = 1$ and

$$350\,\text{cm} = 350\,\text{cm} \times 1 = 350\,\text{cm} \times \frac{1\,\text{m}}{100\,\text{cm}} = \frac{350}{100}\,\text{m} = 3.5\,\text{m}$$

2. From the table, 1 gal = 4 qts. Therefore, $\frac{4\,\text{qts}}{1\,\text{gal}} = 1$ and

$$3\,\text{gal} = 3\,\text{gal} \times 1 = 3\,\text{gal} \times \frac{4\,\text{qts}}{1\,\text{gal}} = 12\,\text{qts}.$$

3. From the table, 1 ft = 12 in; 1 in = 2.54 cm. Therefore,

$$5\,\text{ft} = 5\,\text{ft} \times \frac{12\,\text{in}}{1\,\text{ft}} \times \frac{2.54\,\text{cm}}{1\,\text{in}} = 152.4\,\text{cm}$$

4. From the table, 1000 m = 1 km; 1.609 km = 1 mi. Therefore,

$$1500\,\text{m} = 1500\,\text{m} \times \frac{1\,\text{km}}{1000\,\text{m}} \times \frac{1\,\text{mi}}{1.609\,\text{km}} = 1.5\,\text{km} \times \frac{1\,\text{mi}}{1.609\,\text{km}} = 0.932\,\text{mi}$$

Exercises: