SOLIDS, NETS, AND CROSS SECTIONS

Polyhedra

In this section, we will examine various three-dimensional figures, known as solids. We begin with a discussion of polyhedra.

<u>Polyhedron</u>

A *polyhedron* is a three-dimensional solid with the following properties:

1. A polyhedron is composed entirely of polygons; each of these polygons is known as a *face*.
2. The segment where two polygons intersect is known as an *edge*, and the *edge* is a shared side of the two polygons.
3. The point where three or more edges intersect is known as a *vertex*. The vertices of the polyhedron are the vertices of the polygons.

The plural of polyhedron is *polyhedra*.

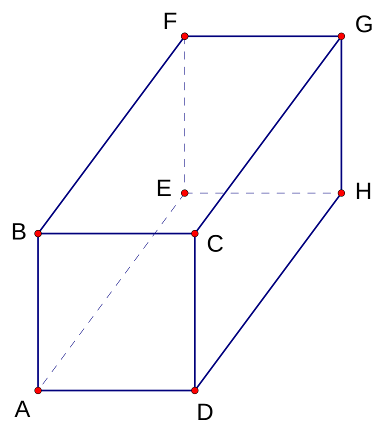
Polyhedra are named according to the number of their faces, as found in the table below. (A very brief and incomplete listing is found here.)

Number of Faces	Name of Polyhedron
4	Tetrahedron
5	Pentahedron
6	Hexahedron
7	Heptahedron
8	Octahedron
9	Enneahedron
10	Decahedron
11	Hendecahedron
12	Dodecahedron
14	Tetradecahedron
20	Icosahedron
24	Icositetrahedron
32	Icosidodecahedron
n	n-hedron

Notice that this naming system is very general, as it only counts the number of faces -- without any regard to the types of polygons that comprise the polyhedron.

Examples
List the faces, edges, and vertices for each of the polyhedra shown below. Then name the polyhedron according to the table above.

1.



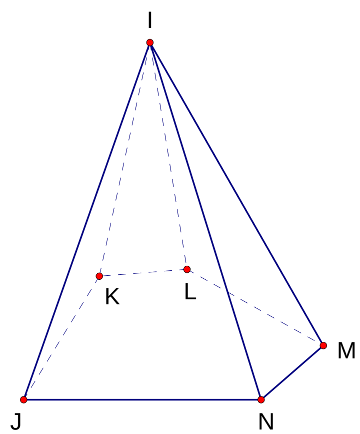
Solution:
The polyhedron has six faces: *ABCD*, *EFGH*, *BFGC*, *AEHD*, *ABFE*, and *DCGH*.

The polyhedron has twelve edges: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{AD}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{EH}$, $\overline{BF}$, $\overline{CG}$, $\overline{AE}$, and $\overline{DH}$.

The polyhedron has eight vertices: *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H*.

Since the polyhedron has six faces, it is called a *hexahedron*.

2.



Solution:
The polyhedron has six faces: ΔIJK, ΔIKL, ΔILM, ΔINM, ΔIJN, and *JKLMN*.

The polyhedron has ten edges: $\overline{IJ}$, $\overline{IK}$, $\overline{IL}$, $\overline{IM}$, $\overline{IN}$, $\overline{JK}$, $\overline{KL}$, $\overline{LM}$, $\overline{MN}$, and $\overline{JN}$.

The polyhedron has six vertices: *I*, *J*, *K*, *L*, *M*, and *N*.

Since the polyhedron has six faces, it is called a *hexahedron*.

Even though the two solids in the examples above are very different from each other, notice that each one can be classified as a hexahedron. In the next section, we will begin to learn about more specific categories of polyhedra.

Exercises

1. Draw a tetrahedron and label its vertices. Then list all of its faces and edges.
2. Draw a pentahedron and label its vertices. Then list all of its faces and edges.

Prisms

<u>Prism</u>

A *prism* is a polyhedron with the following properties:

1. Two congruent polygonal faces lie in parallel planes; each of these two faces is known as a *base*.
2. A segment known as a *lateral edge* joins each vertex of a base with its corresponding vertex on the other base.
3. The remaining faces are parallelograms, and are known as *lateral faces*.

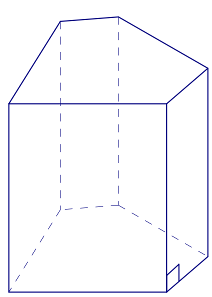
If the lateral edges are perpendicular to the bases, the lateral faces are rectangles and the prism is a *right prism*. Otherwise, the prism is an *oblique prism*.

A prism is named by the shape of its base. If its base is regular (i.e. all sides congruent and all angles congruent), then it is called a *regular prism*.

<u>To draw a prism using transformations</u>:
Draw a base and then translate it onto a plane other than itself. (This image is the other base.) Then connect the corresponding vertices of both bases to form the lateral edges of the prism.

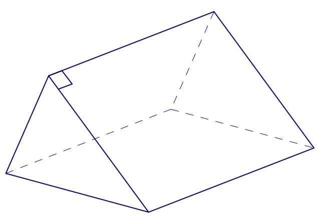
Examples
Name each of the prisms below, and then classify it as right or oblique.

1.



<u>Solution:</u>
This is a pentagonal prism, since the bases are pentagons. It is a right prism, since the lateral edges are perpendicular to the bases.

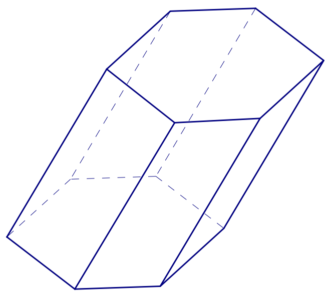
2.



<u>Solution:</u>
This is a triangular prism, since the bases are triangles. (Notice that the bases need only to be congruent and to be in parallel planes;

they don't need to be at the "top" and "bottom" of the diagram.) It is right prism, since the lateral edges are perpendicular to the bases.

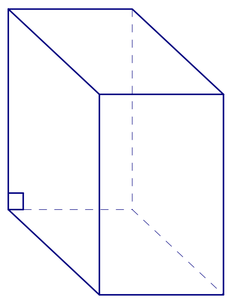
3.



Note: The bases are regular hexagons.

Solution:
This is a regular hexagonal prism, since the bases are regular hexagons. It is an oblique prism, since the lateral edges are not perpendicular to the bases.

4.



Solution:
This is a rectangular prism, since the bases are rectangles. It is a right prism, since the lateral edges are perpendicular to the bases.

Exercises

1. Draw a right octagonal prism. Then state the number of faces (including the bases), edges, lateral faces, lateral edges, and vertices.

2. Draw an oblique regular triangular prism. Then state the number of faces (including the bases), edges, lateral faces, lateral edges, and vertices.

3. Build the following prisms with Polydrons (or a similar manipulative where various polygons can be interlocked together to form solids):
   a. A triangular prism
   b. A cube
   c. A pentagonal prism

Pyramids

<u>Pyramid</u>

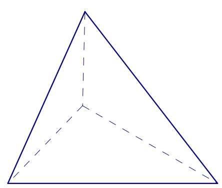
A *pyramid* is a polyhedron with the following properties:

1. One polygon represents the *base*.
2. A point, known as the *apex*, lies on a plane other then the plane of the base itself.
3. A segment known as a *lateral edge* joins each vertex of the base with the apex.
4. The remaining faces are triangles and are known as *lateral faces*.

A pyramid is named by the shape of its base. If its base is regular and its lateral edges are congruent, then it is called a *regular pyramid*.

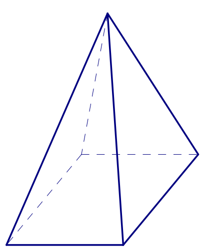
Examples
Name each of the pyramids below.

1.



<u>Solution:</u>
This is a triangular pyramid.

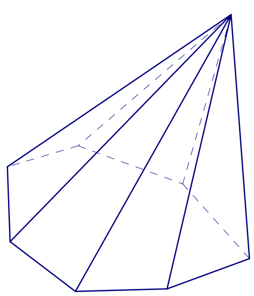
2.



Note: The base is a square.

<u>Solution:</u>
This is a square pyramid.

3.



<u>Solution:</u>
This is a heptagonal pyramid.

Exercises

1. Draw a pentagonal pyramid. Then state the number of faces (including the base), edges, lateral faces, lateral edges, and vertices.

2. Draw a hexagonal pyramid. Then state the number of faces (including the base), edges, lateral faces, lateral edges, and vertices.

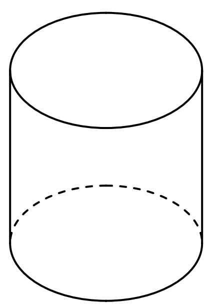
3. Build the following pyramids with Polydrons (or a similar manipulative where various polygons can be interlocked together to form solids):
   a. Triangular pyramid
   b. Square pyramid
   c. Pentagonal pyramid

Cylinders

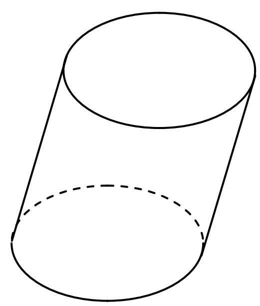
A cylinder is very similar to a prism, in that its two bases are congruent. The *bases*, however, are circles instead of polygons. Because of the curved shape of the base, there are no lateral faces, but instead one smooth *lateral surface*. (If you think of a cylinder like a soup can, then the lateral surface would be represented by the label.)

The segment joining the centers of the two bases is called the *axis* of the cylinder. If the axis is perpendicular to the bases, then the cylinder is a *right cylinder*. If the axis is not perpendicular to the bases, then the cylinder is an *oblique cylinder*.

<u>Diagrams of cylinders:</u>



Right Cylinder



Oblique Cylinder

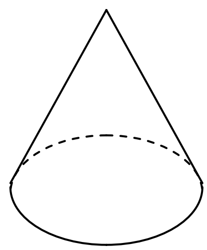
Even though a cylinder is a three-dimensional solid, it cannot be classified as a polyhedron because it is not made up of polygons. (A circle is not a polygon since it is curved rather than composed of segments.)

Cones

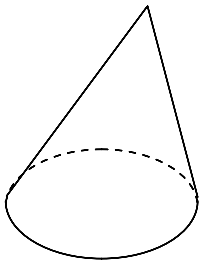
A cone is very similar to a pyramid. Its *base*, however, is a circle rather than a polygon. It has an *apex* just as a pyramid has an apex, but due to the curved shape of the base, there are no lateral faces, but instead a smooth *lateral surface.* (Since a cone is not composed of polygons, it cannot be classified as a polyhedron.)

The segment joining the apex to the center of the base is called the *axis* of the cone. If the axis is perpendicular to the base, then the cone is a *right cone.* If the axis is not perpendicular to the base, then the cylinder is an *oblique cone*.

Diagrams of cones:



Right Cone

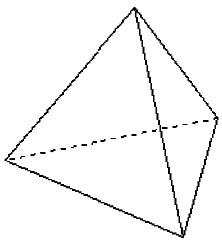


Oblique Cone

Platonic Solids

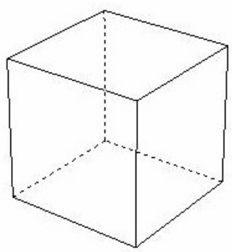
There are five special polyhedra which are known as the Platonic Solids. Each of the Platonic Solids is made up of congruent regular polygons. The five Platonic Solids are listed and illustrated below:

1. The Tetrahedron



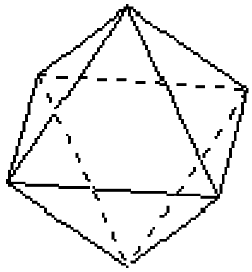
The tetrahedron is made up of four congruent equilateral triangles.

2. The Cube



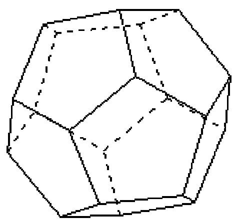
The cube is made up of six congruent squares.

3. The Octahedron



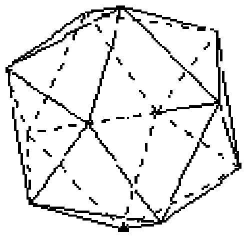
The octahedron is made up of eight congruent equilateral triangles.

4. The Dodecahedron



The dodecahedron is made up of twelve congruent regular pentagons.

5. The Icosahedron



The icosahedron is made up of twenty congruent equilateral triangles.

Exercise

Build the five Platonic Solids with Polydrons (or a similar manipulative where various polygons can be interlocked together to form solids).

Nets

Suppose that we wanted to use a "pattern" to create a solid in the same way that a dressmaker uses a pattern to make a dress. Such a pattern is called a *net*.

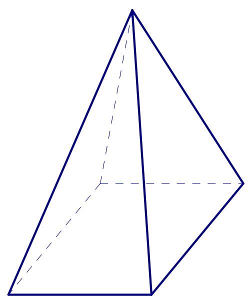
<u>Net</u>

A net is a two-dimensional pattern for a solid.

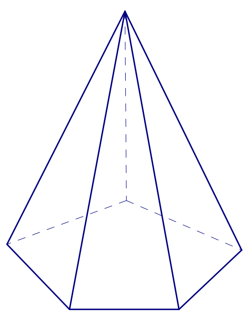
Examples

Imagine being able to "unfold" each of the solids below, and draw a possible net for each solid. Assume that all bases are regular polygons. (If a manipulative such as Polydrons is available, build these solids and then "unfold" them to see what each net looks like.)

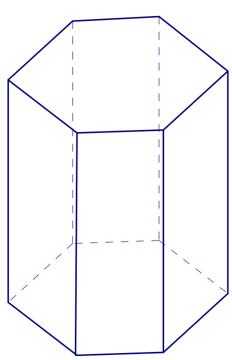
1.



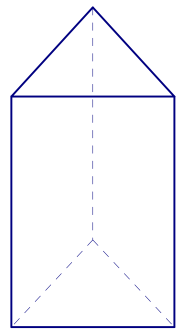
2.



3.



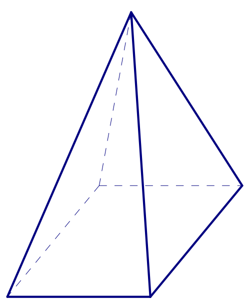
4.

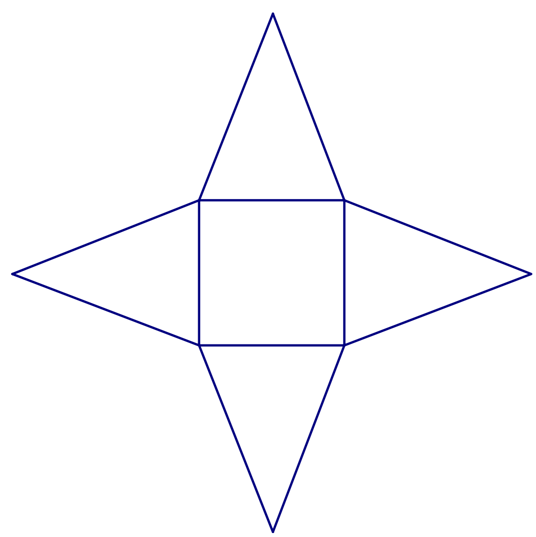


<u>Solutions:</u>
The original solid is shown at the left, with a possible net drawn on the right.

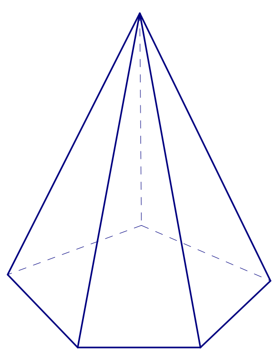
<u>Solid</u> <u>Net</u>

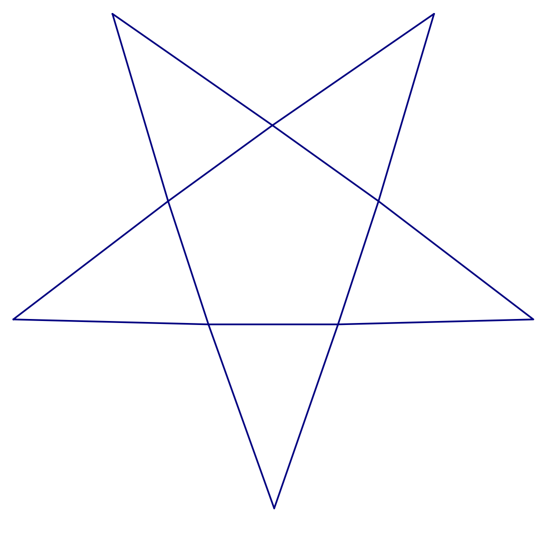
1.





2.



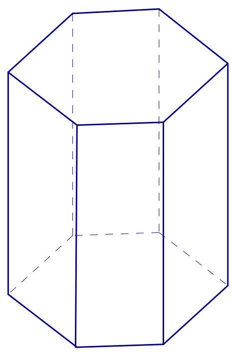


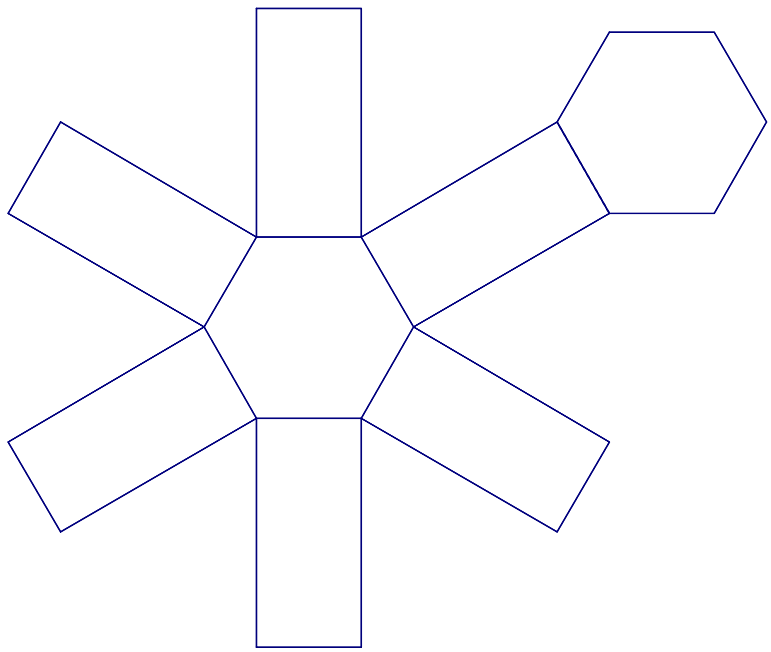
Solutions (continued):
The original solid is shown at the left, with a possible net drawn on the right.

Solid

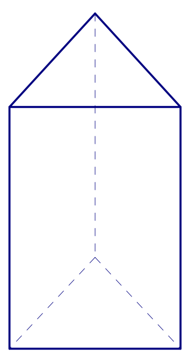
Net

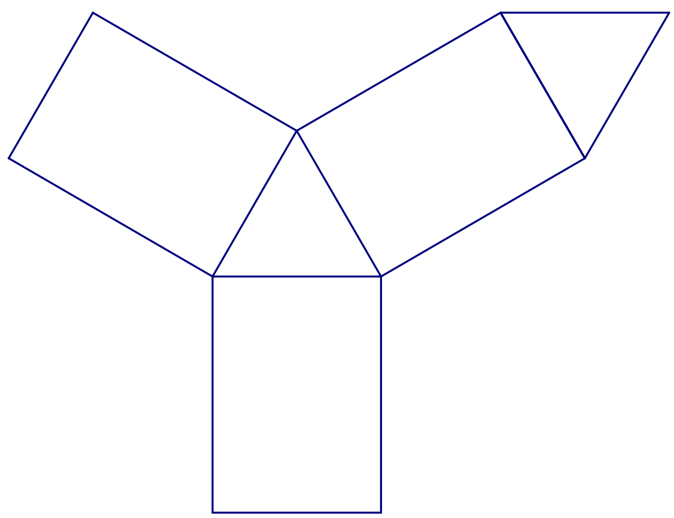
3.





4.





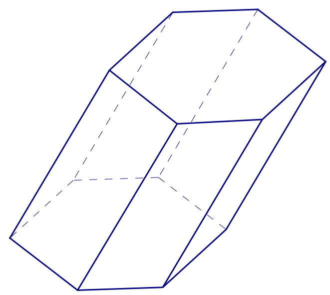
Exercises
Answer the following.

1. Draw a regular right hexagonal pyramid and then draw its net.

2. Draw a rectangular prism and then draw its net.

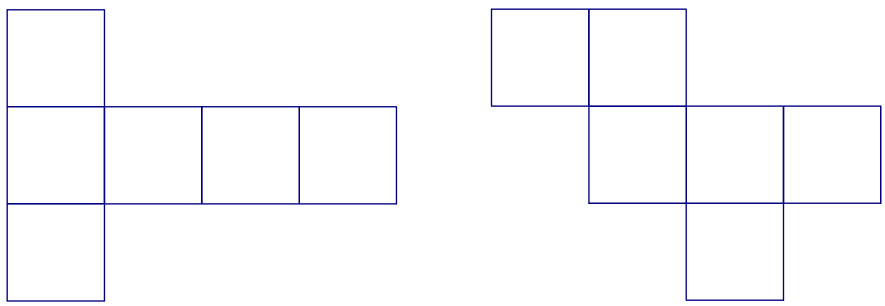
3. Draw a regular right octagonal prism and then draw its net.

4. Draw a net for the following oblique hexagonal prism. (Hint: In an oblique prism, what are the lateral faces made of?)



Note: The bases are regular hexagons.

5. The two nets below represent different nets for the same cube. There are eleven different nets (i.e. nine others) for a cube. Draw as many as you can. (Note: Do not count congruent nets twice; if a net can be rotated or reflected and it looks identical to another one, then the nets are congruent.)



6. Draw a net for a right cylinder. (In your answer, the bases will barely be "connected" to the rest of the net, but this still shows what a cylinder looks like when it is made into a two-dimensional pattern.)

7. Draw a net for a right cone. (As with the cylinder above, the base will barely be "connected" to the rest of the net. This one is difficult; you may want to experiment with a piece of paper and scissors to try to create the lateral surface of a cone.)

Cross Sections

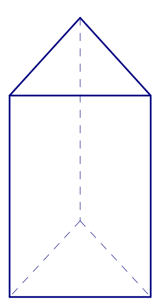
We will now turn our attention to the cross sections of solids. To better understand cross sections, imagine that the solid is made up of moldable clay, and then imagine slicing it with a knife or string. The two-dimensional object seen on the sliced plane of the solid is known as a *cross section*.

Cross sections can be quite complicated, depending on how the object is sliced. For simplicity, we will limit our discussion to the following:

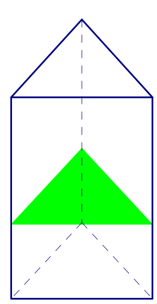
1. All prisms and pyramids in this section have regular polygons as bases.
2. Cross sections will be limited in this section to those which are parallel or perpendicular to a base.
3. The solids in this section will be limited to the following: right prisms, right pyramids (pyramids in which the apex lies directly above the center of the base), right cylinders, and right cones.
4. In cross sections for prisms where the cross section is perpendicular to a base, the “slicing” will be performed through the center of the base, and also through the midpoint of one of the base edges.
5. In cross sections for pyramids where the cross section is perpendicular to the base, the “slicing” will be performed through the apex and also through the midpoint of one of the base edges. (This cross section will then also pass through the center of the base, since each pyramid in this section is a right pyramid with a regular polygon for a base.)

Right Prisms

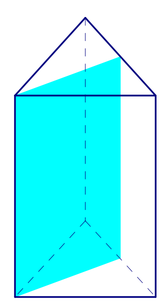
Let us consider the following right prism and then draw some general conclusions.



Initial Diagram



Cross Section Parallel to the Bases



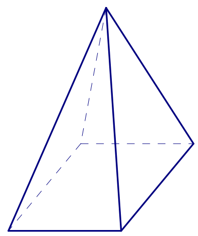
Cross Section Perpendicular to the Bases

We make the following observations which can be generalized to other right prisms:

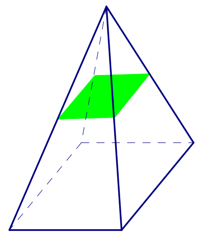
1. In the cross section parallel to the bases, the cross section is congruent to the bases.
2. In the cross section perpendicular to the bases, the cross section is a rectangle.

Right Pyramids

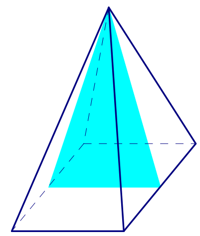
Let us consider the following right pyramid and then draw some general conclusions.



Initial Diagram



Cross Section Parallel
to the Base



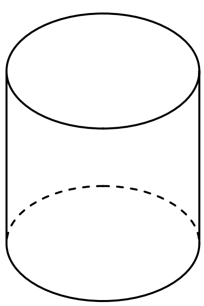
Cross Section Perpendicular
to the Base

We make the following observations which can be generalized to other right pyramids:

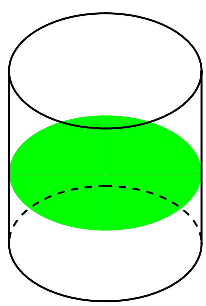
1. In the cross section parallel to the base, the cross section is similar to the base. (In this particular diagram, the base is a square, and any cross section parallel to the base will result in a smaller square.)
2. In the cross section perpendicular to the base, the cross section is a triangle.

Right Cylinders

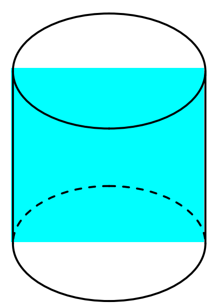
Let us consider the following right cylinder and then draw some general conclusions.



Initial Diagram



Cross Section Parallel
to the Bases



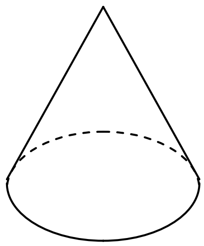
Cross Section Perpendicular
to the Bases

We make the following observations which can be generalized to other right cylinders:

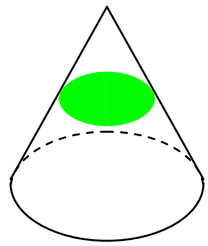
1. In the cross section parallel to the bases, the cross section is a circle which is congruent to the bases.
2. In the cross section perpendicular to the bases, the cross section is a rectangle.

Right Cones

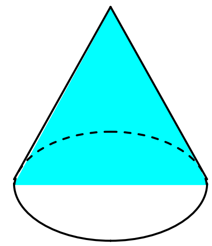
Let us consider the following right cone and then draw some general conclusions.



Initial Diagram



Cross Section Parallel
to the Base



Cross Section Perpendicular
to the Base

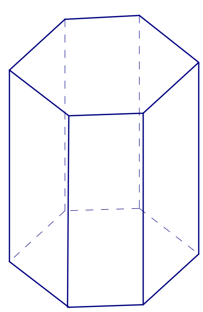
We make the following observations which can be generalized to other right cones:

1. In the cross section parallel to the base, the cross section is another circle which is smaller than the base.
2. In the cross section perpendicular to the base, the cross section is a triangle.

Exercises

For each of the solids below, sketch two cross sections as described in the examples above. One cross section should be parallel to a base, and the other perpendicular to a base. Then identify each of the cross sections with a name (regular pentagon, triangle, rectangle, circle, etc.)

1.



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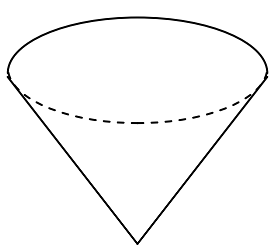

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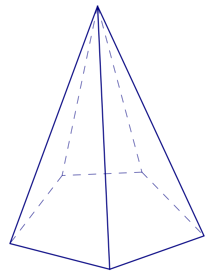

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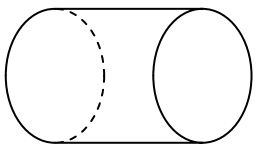

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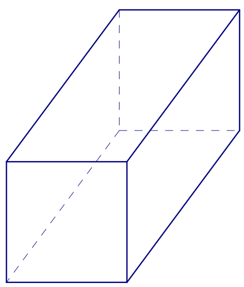
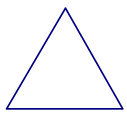
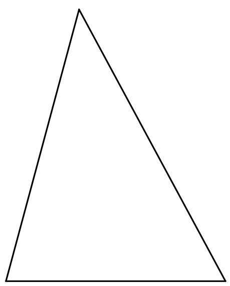


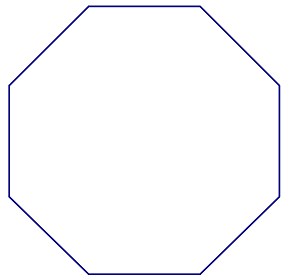
Exercises
For each of the exercises below, sketch a solid which could have the given cross sections. (Some answers may not be unique.)

1. Cross section parallel to a base: Cross section perpendicular to a base:

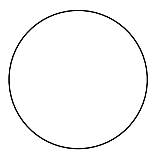


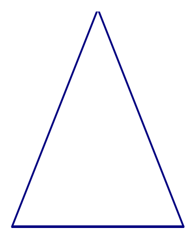


2. Cross section parallel to a base: Cross section perpendicular to a base:

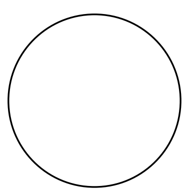


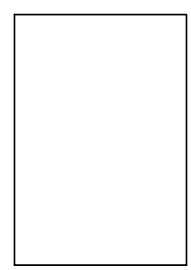
3. Cross section parallel to a base: Cross section perpendicular to a base:





4. Cross section parallel to a base: Cross section perpendicular to a base:





References

Ballew, Pat. Images of the five Platonic Solids. Retrieved April 27, 2004, from http://www.pballew.net/platonic.html.