RIGHT TRIANGLE TRIGONOMETRY

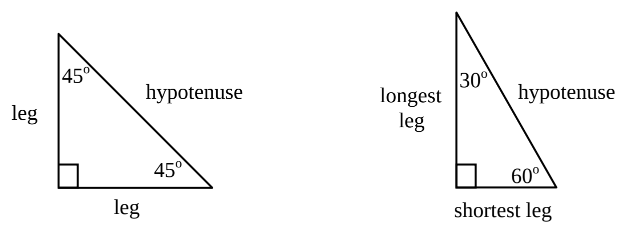
The word Trigonometry can be broken into the parts *Tri*, *gon*, and *metry*, which means "Three angle measurement," or equivalently "Triangle measurement." Throughout this unit, we will learn new ways of finding missing sides and angles of triangles which we would be unable to find using the Pythagorean Theorem alone.

The basic trigonometric theorems and definitions will be found in this portion of the text, along with a few examples, but the reader will frequently be directed to refer to detailed "tutorials" that have numerous examples, explorations, and exercises to complete for a more thorough understanding of each topic.

One comment should be made about our notation for angle measurement. In our study of Geometry, it was standard to discuss the measure of angle A with the notation $m\angle A$. It is a generally accepted practice in higher level mathematics to omit the measure symbol (although there is variation from text to text), so if we are discussing the measure of a 20^{o} angle, for example, we will use the notation $\angle A = 20^{\circ}$ rather than $m\angle A = 20^{\circ}$.

Special Right Triangles

In Trigonometry, we frequently deal with angle measures that are multiples of 30^{o}, 45^{o}, and 60^{o}. Because of this fact, there are two special right triangles which are useful to us as we begin our study of Trigonometry. These triangles are named by the measures of their angles, and are known as 45^{o}-45^{o}-90^{o} triangles and 30^{o}-60^{o}-90^{o} triangles. A diagram of each triangle is shown below:



Tutorial:
For a more detailed exploration of this section along with additional examples and exercises, see the tutorial entitled "Special Right Triangles."

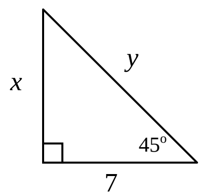
The theorems relating to special right triangles can be found below, along with examples of each.

Theorem:	In a 45°-45°-90° triangle, the legs are congruent, and the length of the hypotenuse is $\sqrt{2}$ times the length of either leg.

Examples

Find *x* and *y* by using the theorem above. Write answers in simplest radical form.

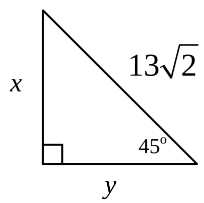
1.



Solution:

The legs of the triangle are congruent, so $x = 7$. The hypotenuse is $\sqrt{2}$ times the length of either leg, so $y = 7\sqrt{2}$.

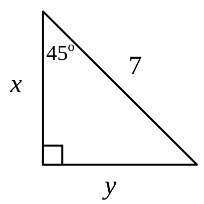
2.



Solution:

The hypotenuse is $\sqrt{2}$ times the length of either leg, so the length of the hypotenuse is $x\sqrt{2}$. We are given that the length of the hypotenuse is $13\sqrt{2}$, so $x\sqrt{2} = 13\sqrt{2}$, and we obtain $x = 13$. Since the legs of the triangle are congruent, $x = y$, so $y = 13$.

3.



Solution:

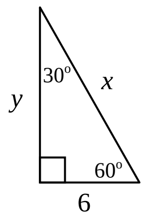
The hypotenuse is $\sqrt{2}$ times the length of either leg, so the length of the hypotenuse is $x\sqrt{2}$. We are given that the length of the hypotenuse is 7, so $x\sqrt{2} = 7$, and we obtain $x = \frac{7}{\sqrt{2}}$. Rationalizing the denominator, $x = \frac{7}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$. Since the legs of the triangle are congruent, $x = y$, so $y = \frac{7\sqrt{2}}{2}$.

Theorem:	In a 30°-60°-90° triangle, the length of the hypotenuse is twice the length of the shortest leg, and the length of the longest leg is $\sqrt{3}$ times the length of the shortest leg.

Examples

Find x and y by using the theorem above. Write answers in simplest radical form.

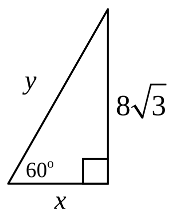
1.



Solution:

The length of the shortest leg is 6. Since the length of the hypotenuse is twice the length of the shortest leg, $x = 2 \cdot 6 = 12$. The length of the longest leg is $\sqrt{3}$ times the length of the shortest leg, so $y = 6\sqrt{3}$.

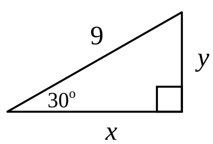
2.



Solution:

The length of the shortest leg is x. Since the length of the longest leg is $\sqrt{3}$ times the length of the shortest leg, the length of the longest leg is $x\sqrt{3}$. We are given that the length of the longest leg is $8\sqrt{3}$, so $x\sqrt{3} = 8\sqrt{3}$, and therefore $x = 8$. The length of the hypotenuse is twice the length of the shortest leg, so $y = 2x = 2 \cdot 8 = 16$.

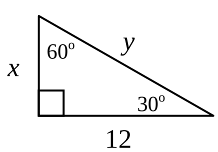
3.



Solution:

The length of the shortest leg is y. Since the length of the hypotenuse is twice the length of the shortest leg, $9 = 2y$, so $y = \frac{9}{2} = 4.5$. Since the length of the longest leg is $\sqrt{3}$ times the length of the shortest leg, $x = y\sqrt{3} = \frac{9}{2}\sqrt{3} = 4.5\sqrt{3}$.

4.



Solution:

The length of the shortest leg is x. Since the length of the longest leg is $\sqrt{3}$ times the length of the shortest leg, the length of the longest leg is $x\sqrt{3}$. We are given that the length of the longest leg is 12, so $x\sqrt{3} = 12$. Solving for x and rationalizing the denominator, we obtain $x = \frac{12}{\sqrt{3}} = 12 \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$. The length of the hypotenuse is twice the length of the shortest leg, so $y = 2x = 2 \cdot 4\sqrt{3} = 8\sqrt{3}$.

Trigonometric Ratios

There are three basic trigonometric ratios which form the foundation of trigonometry; they are known as the sine, cosine and tangent ratios. This section will introduce us to these ratios, and the following sections will help us to use these ratios to find missing sides and angles of right triangles.

Tutorial:
For a more detailed exploration of this section along with additional examples and exercises, see the tutorial entitled "Trigonometric Ratios."

The three basic trigonometric ratios are defined in the table below. The symbol θ, pronounced "theta", is a Greek letter which is commonly used in Trigonometry to represent an angle, and is used in the following definitions. Treat it as you would any other variable.

If θ is an acute angle of a right triangle, then:

Trigonometric Function		**Abbreviation**		**Ratio of the Following Lengths**
The sine of θ	=	$\sin(\theta)$	=	$\dfrac{\text{The leg opposite angle } \theta}{\text{The hypotenuse}}$
The cosine of θ	=	$\cos(\theta)$	=	$\dfrac{\text{The leg adjacent to angle } \theta}{\text{The hypotenuse}}$
The tangent of θ	=	$\tan(\theta)$	=	$\dfrac{\text{The leg opposite angle } \theta}{\text{The leg adjacent to angle } \theta}$

**Note: A useful mnemonic (in very abbreviated form) for remembering the above chart is:*

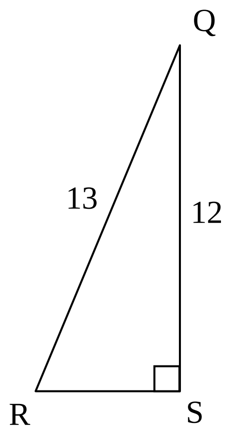
SOH-CAH-TOA

SOH stands for **s**in(θ), **O**pposite, **H**ypotenuse: $\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$

CAH stands for **c**os(θ), **A**djacent, **H**ypotenuse: $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

TOA stands for **t**an(θ), **O**pposite, **A**djacent: $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$

Example
Find the sine, cosine, and tangent ratios for each of the acute angles in the following triangle.



Solution:

We first find the missing length of side *RS*.
Solving the equation $(RS)^2 + 12^2 = 13^2$, we obtain $RS = 5$.

We then find the three basic trigonometric ratios for angle R:

$$\sin R = \frac{\text{The leg opposite angle R}}{\text{The hypotenuse}} = \frac{12}{13}$$

$$\cos R = \frac{\text{The leg adjacent to angle R}}{\text{The hypotenuse}} = \frac{5}{13}$$

$$\tan R = \frac{\text{The leg opposite angle R}}{\text{The leg adjacent to angle R}} = \frac{12}{5}$$

We then find the three basic trigonometric ratios for angle Q:

$$\sin Q = \frac{\text{The leg opposite angle Q}}{\text{The hypotenuse}} = \frac{5}{13}$$

$$\cos Q = \frac{\text{The leg adjacent to angle Q}}{\text{The hypotenuse}} = \frac{12}{13}$$

$$\tan Q = \frac{\text{The leg opposite angle Q}}{\text{The leg adjacent to angle Q}} = \frac{5}{12}$$

Finding Missing Sides of Right Triangles

We will now learn to use the three basic trigonometric ratios to find missing sides of right triangles.

Tutorial:
For a more detailed exploration of this section along with additional examples and exercises, see the tutorial entitled "Using Trigonometry to Find Missing Sides of Right Triangles."

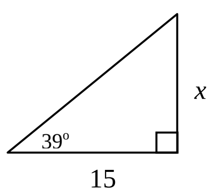
In Trigonometry, there are two basic types of angle measure known as degrees and radians. In this text, we will be using only degree measure, so you should make sure that your calculator is in degree mode. (Refer to the tutorial for more information on how to do this.)

We already have the tools that we have to find missing sides of right triangles; recall the three basic trigonometric ratios from the previous section (in abbreviated form):

SOH-CAH-TOA

$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$

Examples
Find the value of x in each of the triangles below. Round answers to the nearest ten-thousandth.

1. 

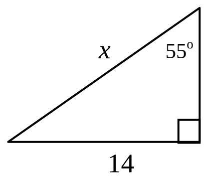
Solution:
Using the 39° angle as our reference angle, x is the length of the opposite leg and 15 is the length of the adjacent leg. Therefore, we will use the tangent ratio:

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan(39^\circ) = \frac{x}{15}$$

$$x = 15 \cdot \tan(39^\circ)$$ *(Enter this into the calculator; make sure first that you are in degree mode.)*

$$x \approx 12.1468$$

2. 

<u>Solution:</u>
Using the 55^o angle as our reference angle, 14 is the length of the opposite leg and *x* is the length of the hypotenuse. Therefore, we will use the tangent ratio:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin(55^\circ) = \frac{14}{x}$$

$$x \cdot \sin(55^\circ) = 14$$

$$x = \frac{14}{\sin(55^\circ)}$$

(Enter this into the calculator; make sure first that you are in degree mode.)

$$x \approx 17.0908$$

Finding Missing Angles of Right Triangles

We will now learn to use the three basic trigonometric ratios to find missing angles of right triangles.

Tutorial:
For a more detailed exploration of this section along with additional examples and exercises, see the tutorial entitled "Using Trigonometry to Find Missing Angles of Right Triangles."

We must first learn how to use the inverse trigonometric function keys on the calculator. Let us consider the following equation:

$$\cos(x) = 0.2934$$

We want to isolate *x*. (Note: We can not divide by "cos" -- it is not a number!) So we need to know the inverse of the cosine function.

On most calculators, this function is labeled "$\cos^{-1}$" can be found in small letters above the "cos" button. (If this is the case, you need to press another button first, since it is not part of the primary keypad. You may, for example, need to press the "2^{nd}" button, and then the "cos" button.)

Back to our example:

$\cos(x) = 0.2934$

$x = \cos^{-1}(0.2934)$ On some calculators, you should press "2nd", then "cos", then 0.2934, then "Enter". On others , you first press 0.2934, then "2nd" then "cos".

$x \approx 72.94^\circ$

If you obtained an answer of 1.27, your calculator is in radian mode instead of degree mode. In this text, we are working strictly with degree measure for angles, so you can keep your calculator in degree mode.

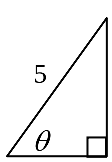
It is important to note that the $\cos^{-1}$ function is NOT a reciprocal function, i.e. $\cos^{-1}(0.2934)$ is NOT the same as $\dfrac{1}{\cos(0.2934)}$. (There is a reciprocal of the cosine function which we will learn about in a later section.)

Examples
Use the inverse functions on your calculator to evaluate the following. Round your answers to the nearest hundredth of a degree.

1. $\sin(x) = 0.2934$ <u>Solution:</u> $x = \sin^{-1}(0.2934) \approx 17.06^\circ$
2. $\tan(\theta) = \frac{15}{11}$ <u>Solution:</u> $\theta = \tan^{-1}\left(\frac{15}{11}\right) \approx 53.75^\circ$

We will now use inverse trigonometric functions to find missing angle measures of right triangles.

Examples
Find the measures of each of the indicated angles in the triangles below. Round your answers to the nearest hundredth.

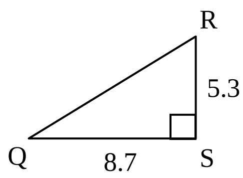
1. 

Find θ.

<u>Solution:</u>
Using θ as our reference angle, 4 is the opposite leg and 5 is the hypotenuse. We therefore use the sine ratio:

$$\sin(\theta) = \frac{4}{5}$$

$$\theta = \sin^{-1}\left(\frac{4}{5}\right) \approx 53.13^\circ$$

2. 

Find $\angle R$.

<u>Solution:</u>
Using $\angle R$ as our reference angle, 8.7 is the opposite leg and 5.3 is the adjacent leg. We therefore use the tangent ratio:

$$\tan(R) = \frac{8.7}{5.3}$$

$$\angle R = \tan^{-1}\left(\frac{8.7}{5.3}\right) \approx 58.65^\circ$$

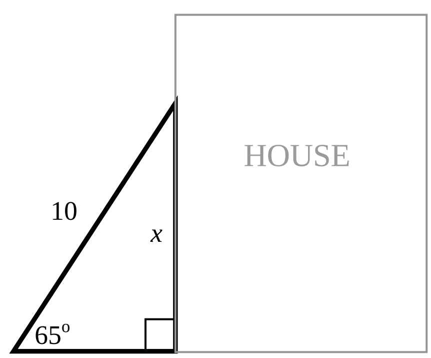
Applications of Right Triangle Trigonometry

Now that we know how to use Trigonometry to find missing sides and angles of right triangles, let us apply this knowledge to some real-world problems. (Note: There is no tutorial for this section since it is simply an application of previous material, but there are additional exercises to complete at the end of the section.)

Examples

1. A ten-foot ladder is leaned against the side of a house in such a way that it makes an angle of 65^o with the ground. How high up the house does the ladder reach? (Round your answer to the nearest tenth.)

   First, draw a picture to represent the given situation.
   We will let $x =$ the height from the top of the ladder to the ground.



   Next, we determine the trigonometric ratio that we can use to solve the problem. If we allow the 65^o angle to be the reference angle, then x is the opposite side and 10 is the hypotenuse. We therefore set up an equation using the sine ratio:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin(65^\circ) = \frac{x}{10}$$

$$x = 10\sin(65^\circ) \approx 10(0.9063) \approx 9.1$$

Finally, answer the question and include units:

The ladder reaches a height of 9.1 feet above the ground.

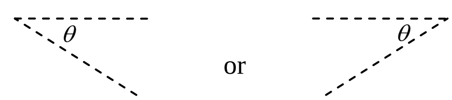
2. A camper is hiking and is standing on top of a 400 foot cliff enjoying the view. He looks down and views a bear at a 37^{o} angle of depression. How far is the bear from the base of the cliff? (Round your answer to the nearest foot, and disregard the height of the camper in your calculations.)

First, we need to clarify the terminology "angle of depression"…

<u>Angles of Depression and Angles of Elevation</u>

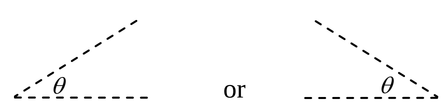
An <u>angle of depression</u> θ is an angle measured downwards from a horizontal line.

Diagrams:

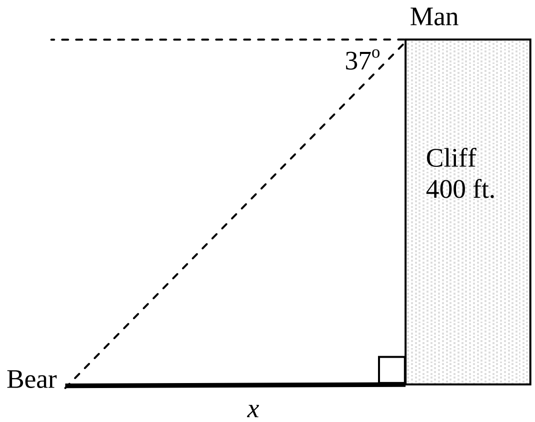


An <u>angle of elevation</u> θ is an angle measured upwards from a horizontal line.

Diagrams:

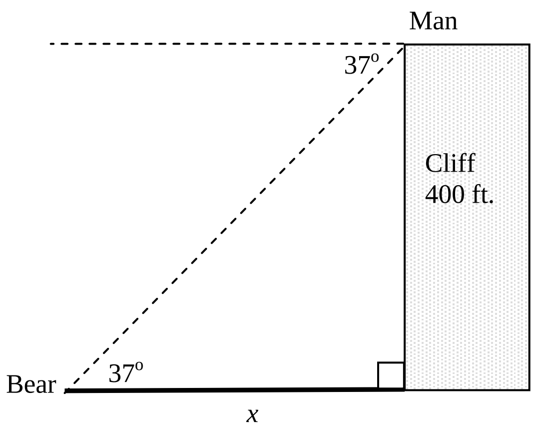


Next, draw a picture to represent the given situation.
We will let $x =$ the bear's distance from the base of the cliff.



There are a variety of ways in which to set up this problem. One method will be shown in detail, and other methods will then be discussed briefly.

Since the two horizontal lines in the picture are parallel, there is another alternate interior angle which is also 37°, as shown below:



Using the 37° angle next to the bear as our reference angle (which, by the way, is an angle of elevation from the bear to the top of the cliff), the 400 foot cliff is the opposite leg and the x is the adjacent leg. Therefore, we will use the tangent ratio:

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\tan(37^\circ) = \frac{400}{x}$$

$$x \cdot \tan(37^\circ) = 400$$

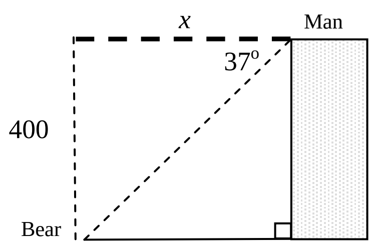
$$x = \frac{400}{\tan(37^\circ)} \approx \frac{400}{0.7536} \approx 531$$

(Note: In reality, the angle of elevation is measured from the man's eyes, which is slightly higher than the top of the cliff, so this answer is not exact – but is a close approximation. This detail was ignored in order to simplify our example.)

Finally, answer the question and include units:

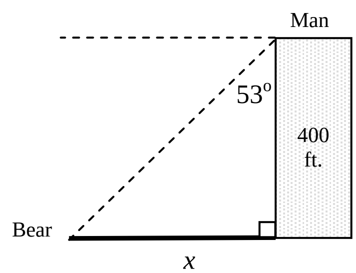
The bear is approximately 531 feet from the base of the cliff.

Two other ways of setting up the problem are shown briefly below, and of course lead to the same solution:



$$\tan(37^\circ) = \frac{400}{x}$$

... $x \approx 531$



$$\tan(53^\circ) = \frac{x}{400}$$

... $x \approx 531$

Exercises:

For each of the following problems, draw a picture to represent the given situation, and then solve the problem.

1. A girl is flying a kite and lets out 250 feet of string. If she sights the kite at a 43^o angle of elevation, what is the height of the kite? (Round your answer to the nearest tenth, and disregard the height of the girl in your calculations.)

2. A ramp is attached to a loading dock, and the ramp makes a 24^o angle with the ground. If the loading dock is 6 feet high, how long is the ramp? (Round your answer to the nearest tenth.)

3. The Washington Monument is 555 feet tall. If an observer is standing 300 feet from the base of the monument, find the angle of elevation from the viewpoint of the observer as he sights the top of the monument. (Round your answer to the nearest tenth, and ignore the height of the observer in your calculations.)

Reciprocal Trigonometric Ratios

We will now learn three ratios which are very useful in higher level mathematics; they are known as the reciprocal trigonometric ratios. (Note: There is no tutorial for this section since it is simply an application of previous material, but there are additional exercises to complete at the end of the section.)

First, let us review the three basic trigonometric ratios that we have learned so far:

If θ is an acute angle of a right triangle, then:

Trigonometric Function		Abbreviation		Ratio of the Following Lengths
The sine of θ	=	$\sin(\theta)$	=	$\dfrac{\text{The leg opposite angle } \theta}{\text{The hypotenuse}}$
The cosine of θ	=	$\cos(\theta)$	=	$\dfrac{\text{The leg adjacent to angle } \theta}{\text{The hypotenuse}}$
The tangent of θ	=	$\tan(\theta)$	=	$\dfrac{\text{The leg opposite angle } \theta}{\text{The leg adjacent to angle } \theta}$

There are three other trigonometric ratios called the reciprocal trigonometric ratios, and they are defined as follows:

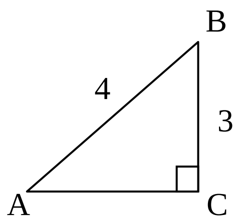
If θ is an acute angle of a right triangle, then:

Trigonometric Function		Abbreviation		Reciprocal Relationship		Ratio of the Following Lengths
The cosecant of θ	=	$\csc(\theta)$	=	$\dfrac{1}{\sin(\theta)}$	=	$\dfrac{\text{The hypotenuse}}{\text{The leg opposite angle } \theta}$
The secant of θ	=	$\sec(\theta)$	=	$\dfrac{1}{\cos(\theta)}$	=	$\dfrac{\text{The hypotenuse}}{\text{The leg adjacent to angle } \theta}$
The cotangent of θ	=	$\cot(\theta)$	=	$\dfrac{1}{\tan(\theta)}$	=	$\dfrac{\text{The leg adjacent to angle } \theta}{\text{The leg opposite angle } \theta}$

Although these ratios are used a great deal in trigonometry (as we will see in the next unit), we have waited until now to introduce them because they were not needed to solve for missing angles and sides of right triangles. Notice that there are no keys on the calculator for cosecant, secant and cotangent – the $\sin^{-1}$ key, for example, is the inverse sine function – NOT the cosecant function.

Example

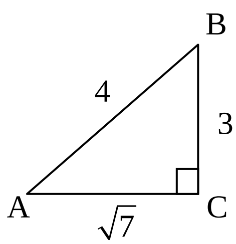
Evaluate the six trigonometric ratios of each acute angle in the triangle below. Write all answers in simplest radical form.



Solution:

First, find the missing side using the Pythagorean Theorem:

$$(AC)^2 + 3^2 = 4^2$$
$$(AC)^2 + 9 = 16$$
$$(AC)^2 = 7$$
$$AC = \sqrt{7}$$



Now we will find the six trigonometric ratios for angle A. (The definitions of the trigonometric ratios are written in abbreviated form.)

$$\sin(A) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{3}{4}$$

$$\csc(A) = \frac{1}{\sin(A)} = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{4}{3}$$

$$\cos(A) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\sqrt{7}}{4}$$

$$\sec(A) = \frac{1}{\cos(A)} = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{4}{\sqrt{7}} = \frac{4}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$$

$$\tan(A) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$

$$\cot(A) = \frac{1}{\tan(A)} = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{\sqrt{7}}{3}$$

Next, we will find the six trigonometric ratios for angle B:

$\sin(B) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\sqrt{7}}{4}$

$\csc(B) = \frac{1}{\sin(B)} = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{4}{\sqrt{7}} = \frac{4}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$

$\cos(B) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{3}{4}$

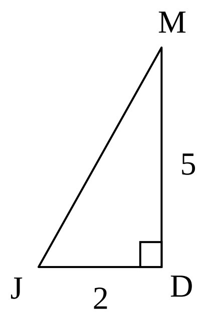
$\sec(B) = \frac{1}{\cos(B)} = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{4}{3}$

$\tan(B) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\sqrt{7}}{3}$

$\cot(B) = \frac{1}{\tan(B)} = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$

Exercises

Find the indicated trigonometric ratios for the triangle below. Write all answers in simplest radical form.



1. $\sin(J)$
2. $\cos(J)$
3. $\tan(J)$
4. $\sin(M)$
5. $\cos(M)$
6. $\tan(M)$
7. $\csc(J)$
8. $\sec(J)$
9. $\cot(J)$
10. $\csc(M)$
11. $\sec(M)$
12. $\cot(M)$