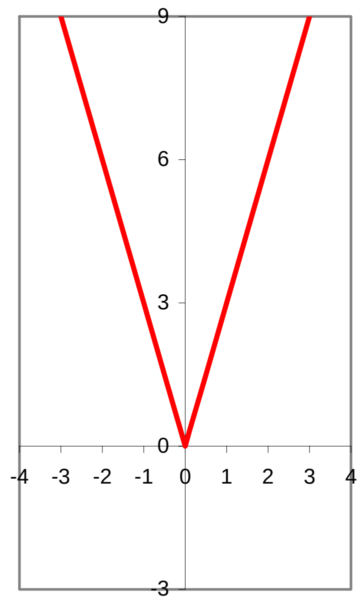
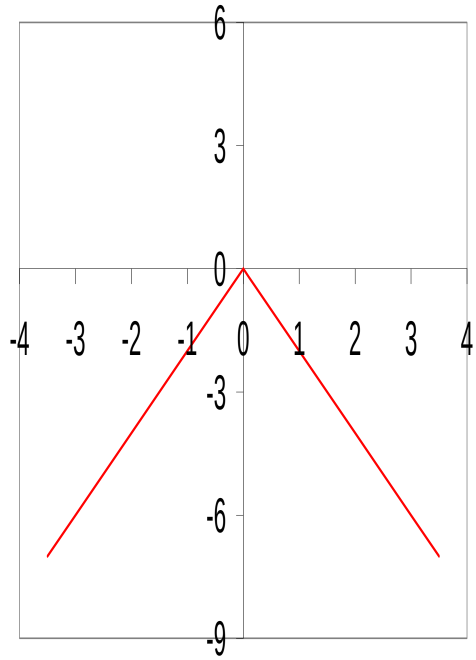
NONLINEAR FUNCTIONS
A. Absolute Value

Exercises:

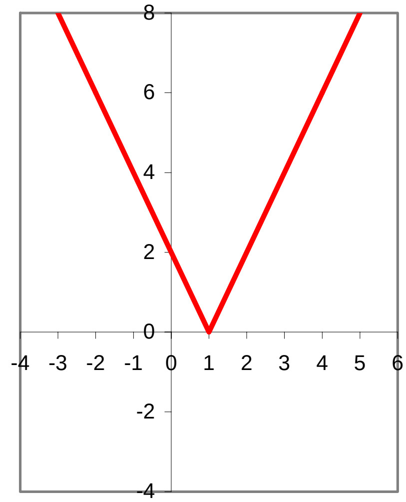
1. We need to scale the graph of $Q(x)=|x|$ by the factor of 3 to get the graph of $f(x)=3|x|$. The graph is given below.



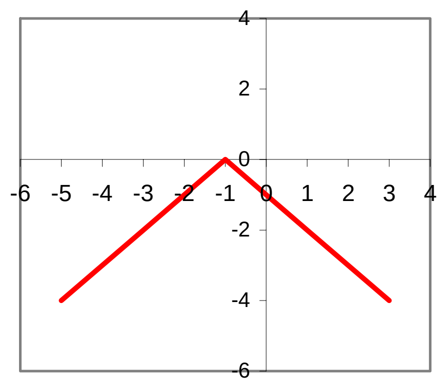
2. We need to scale the graph of $Q(x)=|x|$ by the factor of -2 to get the graph of $f(x)=-2|x|$. The graph is given below.



3. To get the graph of $f(x) = 2|x-1|$ by using $Q(x) = |x|$, firstly scale $Q(x) = |x|$ by a factor of 2 and then shift the graph 1 unit in the x direction.

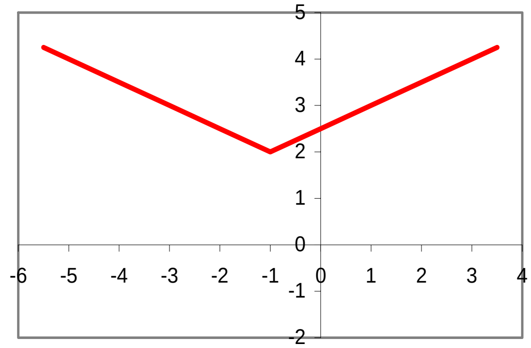


4. To draw the graph of $G(x) = -|x+1|$ by using $Q(x) = |x|$, firstly reflect the graph of $Q(x) = |x|$ across the x axis and then shift the new graph -1 unit in the x direction.

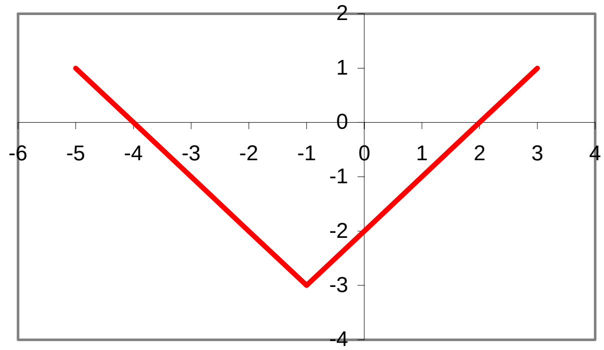


5. The graph of $R(x) = \frac{1}{2}|x-1| + 2$ can be obtained from $Q(x) = |x|$ by

- Scaling the graph of $Q(x) = |x|$ by a factor of $1/2$
- Shifting the new graph 1 unit in x direction and 2 units in y direction.

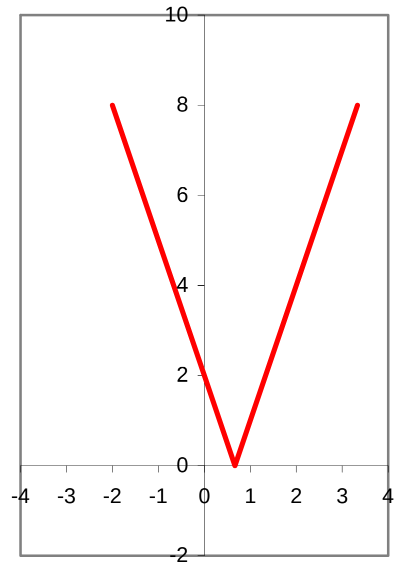


6. The graph of $T(x) = |x+1| - 3$ can be obtained from $Q(x) = |x|$ by shifting the graph of $Q(x) = |x|$ first -1 unit in x direction and then -3 units in y direction.



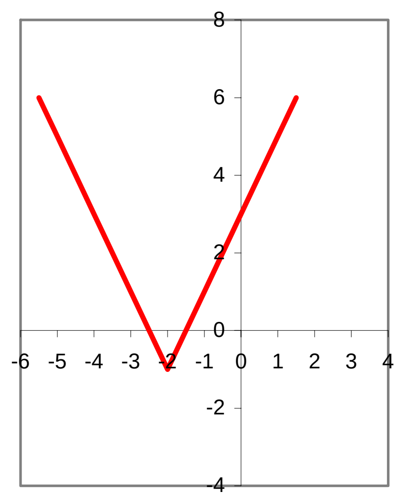
7. The graph of $H(x) = |3x-2| = \left|3\left(x - \frac{2}{3}\right)\right| = 3\left|x - \frac{2}{3}\right|$ can be obtained from $Q(x) = |x|$ by

- Scaling the graph of $Q(x) = |x|$ by a factor of 3
- Shifting the new graph $2/3$ units in x direction.



8. The graph of $H(x) = |2x+4| - 1 = |2(x+2)| - 1 = 2|x+2| - 1$ can be obtained from $Q(x) = |x|$ by

- Scaling the graph of $Q(x) = |x|$ by a factor of 2
- Shifting the new graph -2 units in x direction and -1 unit in y direction.



9. $G(x) = -|x+1| = \begin{cases} -(x+1) & \text{, if } x+1 \geq 0 \\ -(-(x+1)) & \text{, if } x+1 < 0 \end{cases} = \begin{cases} -x-1 & \text{, if } x+1 \geq 0 \\ x+1 & \text{, if } x+1 < 0 \end{cases}$ and

$x+1 \geq 0 \iff x \geq -1$.

Hence, $G(x)$ can be written as:

$$G(x) = \begin{cases} -x-1 & \text{, if } x \geq -1 \\ x+1 & \text{, if } x < -1 \end{cases}.$$

10. $T(x)=|x+1|-3=\begin{cases}(x+1)-3 & \text{, if } x+1\ge 0\\ -(x+1)-3 & \text{, if } x+1<0\end{cases}=\begin{cases}x-2 & \text{, if } x+1\ge 0\\ -x-4 & \text{, if } x+1<0\end{cases}$ and

$x+1\ge 0 \iff x\ge -1$. Then,

$$T(x)=\begin{cases}x-2 & \text{, if } x\ge -1\\ -x-4 & \text{, if } x<-1\end{cases}.$$

11. $R(x)=\frac{1}{2}|x-1|+2=\begin{cases}\frac{1}{2}(x-1)+2 & \text{, if } x-1\ge 0\\ -\frac{1}{2}(x-1)+2 & \text{, if } x-1<0\end{cases}=\begin{cases}\frac{1}{2}x+\frac{3}{2} & \text{, if } x-1\ge 0\\ -\frac{1}{2}x+\frac{5}{2} & \text{, if } x-1<0\end{cases}$ and

$x-1\ge 0 \iff x\ge 1$. So,

$$R(x)=\begin{cases}\frac{1}{2}x+\frac{3}{2} & \text{, if } x\ge 1\\ -\frac{1}{2}x+\frac{5}{2} & \text{, if } x<1\end{cases}.$$

12. $H(x)=|3x-2|=\begin{cases}3x-2 & \text{, if } 3x-2\ge 0\\ -(3x-2) & \text{, if } 3x-2<0\end{cases}=\begin{cases}3x-2 & \text{, if } 3x-2\ge 0\\ -3x+2 & \text{, if } 3x-2<0\end{cases}$ and

$3x-2\ge 0 \iff 3x\ge 2 \iff x\ge \frac{2}{3}$. Thus,

$$H(x)=\begin{cases}3x-2 & \text{, if } x\ge 2/3\\ -3x+2 & \text{, if } x<2/3\end{cases}.$$

13. $M(x)=|2x+4|-1=\begin{cases}(2x+4)-1 & \text{, if } 2x+4\ge 0\\ -(2x+4)-1 & \text{, if } 2x+4<0\end{cases}=\begin{cases}2x+3 & \text{, if } 2x+4\ge 0\\ -2x-5 & \text{, if } 2x+4<0\end{cases}$ and

$2x+4\ge 0 \iff 2x\ge -4 \iff x\ge -2$. Hence,

$$M(x)=\begin{cases}2x+3 & \text{, if } x\ge -2\\ -2x-5 & \text{, if } x<-2\end{cases}.$$

B. Polynomial Functions

Exercises:

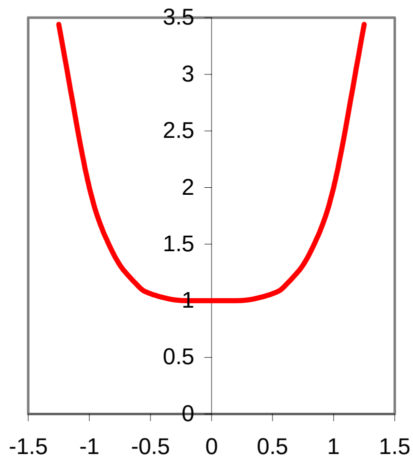
1. $P(x)=-x^4+8=-x^4+0x^3+0x^2+0x+8$ is a polynomial of degree 4.

2. $F(x)=-3x^3+2x^2+12=-3x^3+2x^2+0x+12$ is a polynomial of degree 3.

3. $f(x)=3x^2-2x+1=ax^2+bx+c$ is a second degree polynomial

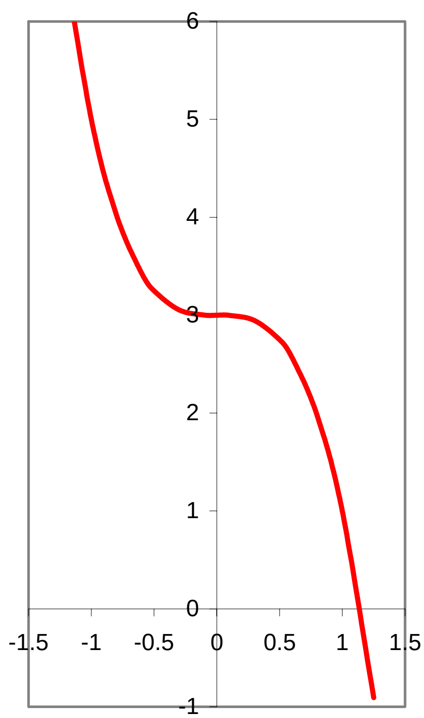
4. $h(x)=-7x^5+3x^3-2x+1=-7x^5+0x^4+3x^3+0x^2-2x+1$ is a polynomial of degree 5.

5. To graph the polynomial function $f(x) = -x^4 + 1$, we need to reflect the graph of $f(x) = x^4$ across the *x* axis and then shift the new graph 1 unit in *y* direction.



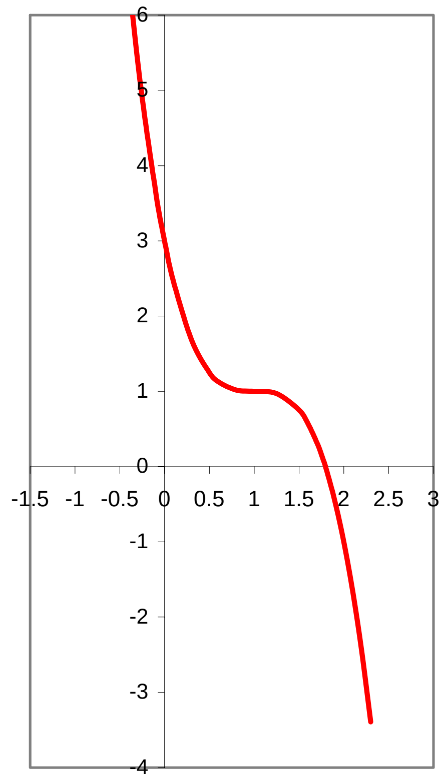
6. To graph $g(x) = -2x^3 + 3$, we need to

- Reflect the graph of $f(x) = x^3$ across the *x* axis
- Scale the new graph by a factor of 2
- Shift the graph 3 units in *y* direction.



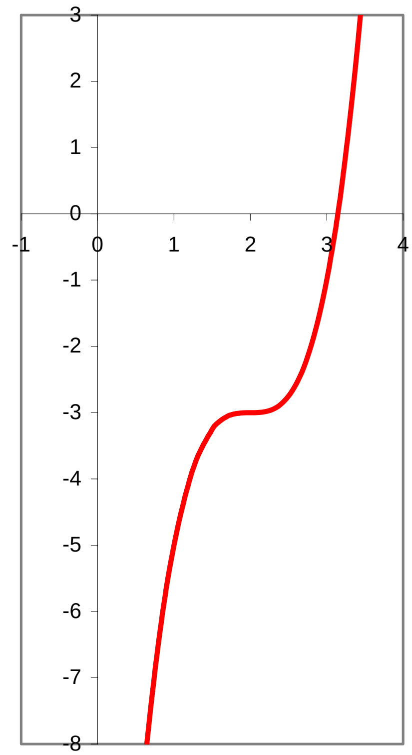
7. To draw the graph of $h(x) = -2(x-1)^3 + 1$, we need to:

- Reflect the graph of $f(x) = x^3$ across the x axis
- Scale the new graph by a factor of 2
- Shift the graph 1 unit in x direction and 1 unit in y direction.



8. To draw the graph of $F(x) = 2(x-2)^3 - 3$, we need to:

- Scale the graph of $f(x) = x^3$ by a factor of 2
- Shift the graph 2 units in x direction and -3 units in y direction.



9. To obtain the graph of $F(x) = -2(x+3)^7 - 11$ from the graph of $g(x) = x^7$;

- Reflect the graph of $g(x) = x^7$ across the *x* axis
- Scale the graph by a factor of 2
- Shift the new graph -3 units in *x* direction and -11 units in *y* direction.

10. To obtain the graph of $F(x) = 3(x-2)^{12} + 5$ from the graph of $g(x) = x^{12}$;

- Scale the graph of $g(x) = x^{12}$ by a factor of 3
- Shift the new graph 2 units in *x* direction and -5 units in *y* direction.

11. To obtain the graph of $F(x) = (4x+1)^7 + 2 = \left(4\left(x + \frac{1}{4}\right)\right)^7 + 2 = 4^7\left(x + \frac{1}{4}\right)^7 + 2$ from the graph of $g(x) = x^7$;

- Scale the graph of $g(x) = x^7$ by a factor of 4^7
- Shift the new graph $-1/4$ units in *x* direction and -2 units in *y* direction.

C. Rational Functions

Exercises:

1. $f(x)=\dfrac{x}{x^2-4}$ has vertical asymptotes when $x^2-4=0 \Leftrightarrow x^2=4 \Leftrightarrow x=\pm 2$ (Notice that the numerator is not 0 when $x=\pm 2$).
Since the degree of the numerator is smaller than the degree of the denominator, the horizontal asymptote is $y=0$.

2. The vertical asymptote of $g(x)=\dfrac{2x+1}{x-6}$ is $x=6$ since $x-6=0 \Leftrightarrow x=6$ (the numerator is not zero at this value).
The degrees of the numerator and the denominator are equal, so $y=\dfrac{2}{1}=2$ is the horizontal asymptote of $g(x)=\dfrac{2x+1}{x-6}$.

3. To find the vertical asymptote(s) of $h(x)=\dfrac{3x^2+1}{x^2-3x+2}$ we need to solve the equation. $x^2-3x+2=0$. We have:

$$x^2-3x+2=(x-1)(x-2)=0 \Leftrightarrow x=1 \text{ or } x=2.$$

Since the numerator is not 0 at these values of x, the vertical asymptotes are $x=1$ and $x=2$.

The numerator and the denominator are same degree polynomials; $y=\dfrac{3}{1}=3$ is the horizontal asymptote.

4. $R(x)=\dfrac{2x+1}{x^2+4x-12}$ has vertical asymptotes if $x^2+4x-12=0$ has solutions. Since

$$x^2+4x-12=(x+6)(x-2)=0 \Leftrightarrow x=-6 \text{ or } x=2,$$

the vertical asymptotes are $x=-6$ and $x=2$ (the numerator is not 0 for these values).

The horizontal asymptote is $y=0$ since the degree of the numerator is smaller than the degree of the denominator.

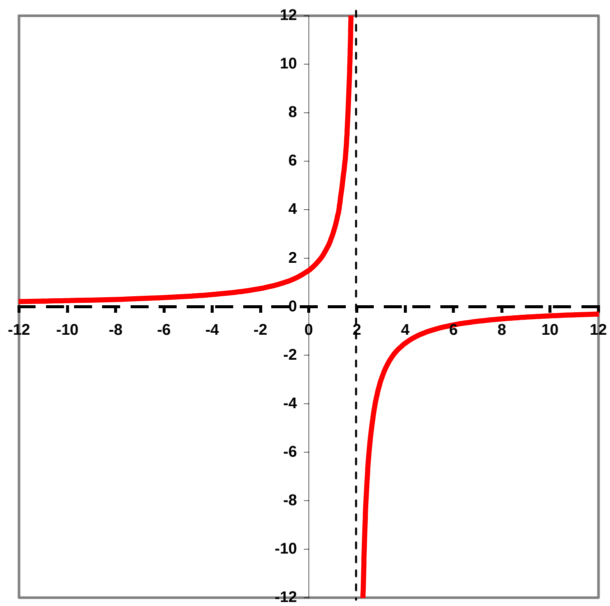
5. *i*) $f(x)=\dfrac{x}{x^2-4}$;
Domain: all values of x except $x=\pm 2$.
The x intercept is $x=0$ since $f(0)=0$.
The y intercept is $y=0$.

ii) $g(x)=\dfrac{2x+1}{x-6}$;
Domain: all values of x except $x=6$.
The x intercept is $x=-1/2$ since $g(-1/2)=0$.
The y intercept is $y=-1/6$ since $g(0)=-1/6$.

iii) $h(x)=\dfrac{3x^2+1}{x^2-3x+2}$;
Domain: all values of *x* except $x=-6$ and $x=2$.
There is no *x* intercepts since $3x^2+1=0$ has no solutions.
The *y* intercept is $y=1/2$ since $f(0)=1/2$.

iv) $R(x)=\dfrac{2x+1}{x^2+4x-12}$;
Domain: all values of *x* except $x=1$ and $x=2$.
The *x* intercept is $x=-1/2$ since $R(-1/2)=0$.
The *y* intercept is $y=-1/12$ since $f(0)=-1/12$.

6. The graph of $h(x)=\dfrac{-3}{x-2}$ is given below.

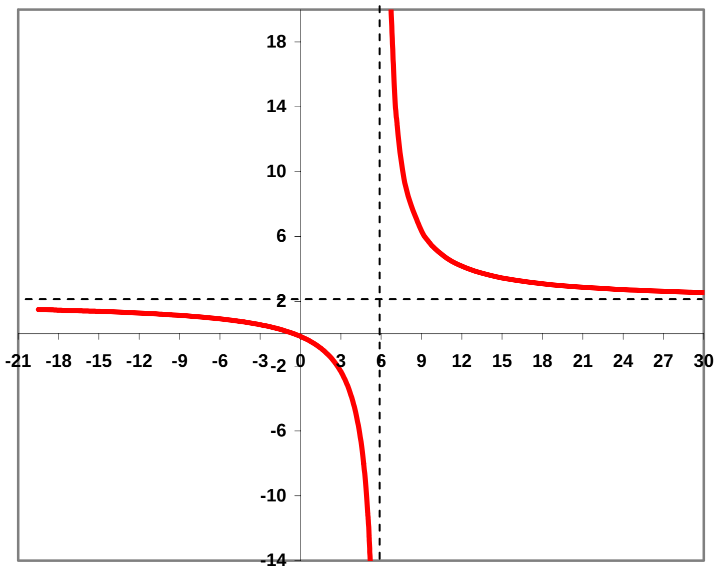


The vertical asymptote of $h(x)=\dfrac{-3}{x-2}$ is $x=2$ and the horizontal asymptote is $y=0$.

To get the graph of $h(x)=\dfrac{-3}{x-2}$ from $g(x)=\dfrac{1}{x}$, we need to:

- Reflect the graph of $g(x)=\dfrac{1}{x}$ across the *x* axis
- Scale the new graph by a factor of 3
- Shift the graph 2 units in *x* direction.

7. The graph of $g(x)=\dfrac{2x+1}{x-6}$ is given below.

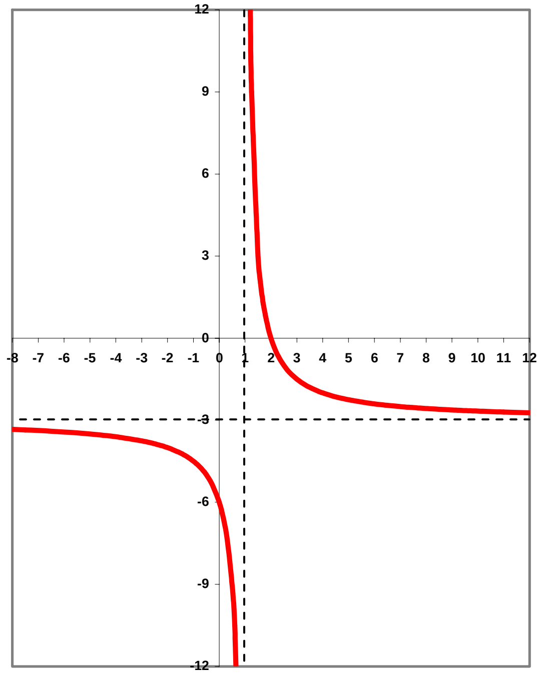


The vertical asymptote of $g(x)=\dfrac{2x+1}{x-6}$ is $x=6$ and the horizontal asymptote is $y=2$.

To get the graph of $g(x)=\dfrac{2x+1}{x-6}=2+\dfrac{13}{x-6}$ from $R(x)=\dfrac{1}{x}$, we need to:

- Scale the graph of $R(x)=\dfrac{1}{x}$ by a factor of 13
- Shift the graph 6 units in *x* direction and 2 units in *y* direction.

8. The graph of $f(x)=\dfrac{-3x+6}{x-1}$ is given below.



The vertical asymptote of $f(x)=\dfrac{-3x+6}{x-1}$ is $x=1$ and the horizontal asymptote is $y=-3$.

To get the graph of $f(x)=\dfrac{-3x+6}{x-1}=-3+\dfrac{3}{x-1}$ from $g(x)=\dfrac{1}{x}$, we need to:

- Scale the graph of $g(x)=\dfrac{1}{x}$ by a factor of 3
- Shift the graph 1 unit in x direction and -3 units in y direction.

D. Exponential Functions

Exercises:

1. The horizontal asymptote for the graph of $g(x)=2^x-3$ is $y=-3$.
The y intercept is $y=-2$ since $g(0)=2^0-3=1-3=-2$.

2. We have $f(x)=\left(\dfrac{1}{3}\right)^{-x}=\left(3^{-1}\right)^{-x}=(3)^{(-1)(-x)}=3^x$, which means that $f(x)=\left(\dfrac{1}{3}\right)^{-x}$ and $f(x)=3^x$ have the same graph.

3. The horizontal asymptote for the graph of $f(x)=\left(\frac{1}{4}\right)^{-x}+5=\left(4^{-1}\right)^{-x}+5=4^{x}+5$ is $y=5$.

The y intercept is $y=6$ since $g(0)=4^{0}+5=6$.

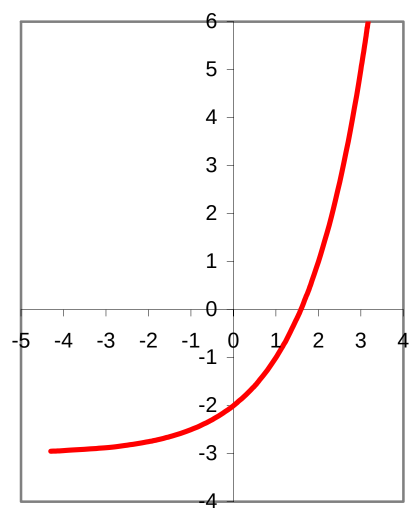
4. To get the graph of $g(x)=2^{x}-3$ from $f(x)=2^{x}$, we need to shift the graph of $f(x)=2^{x}$ -3 units in y direction.

5. To get the graph of $g(x)=\left(\frac{1}{3}\right)^{x+3}-7$ from $f(x)=\left(\frac{1}{3}\right)^{x}$, we need to shift the graph of $f(x)=\left(\frac{1}{3}\right)^{x}$

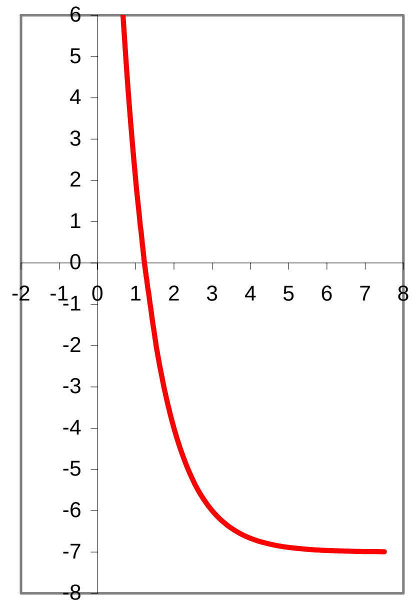
- -3 units in x direction
- 7 units in y direction.

6. To get the graph of $g(x)=5^{2x}-1=\left(5^{2}\right)^{x}-1=25^{x}-1$ from $f(x)=25^{x}$, we need to shift the graph of $f(x)=25^{x}$ -1 unit in y direction.

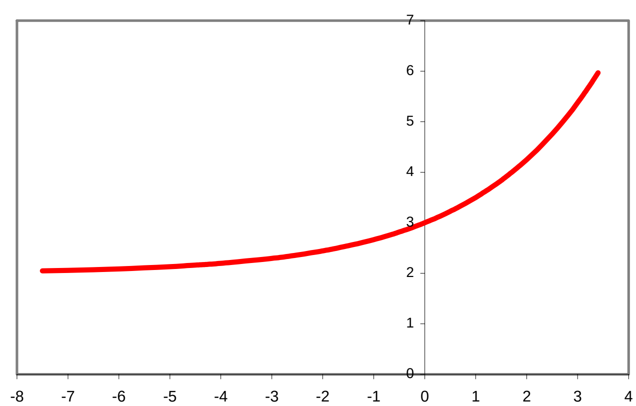
7. To graph $g(x)=2^{x}-3$, shift the graph of $f(x)=2^{x}$ -3 units in y direction.



8. To get the graph of $g(x)=\left(\frac{1}{3}\right)^{x+3}-7$ we need to shift the graph of $f(x)=\left(\frac{1}{3}\right)^{x}$ -3 units in x direction and -7 units in y direction.



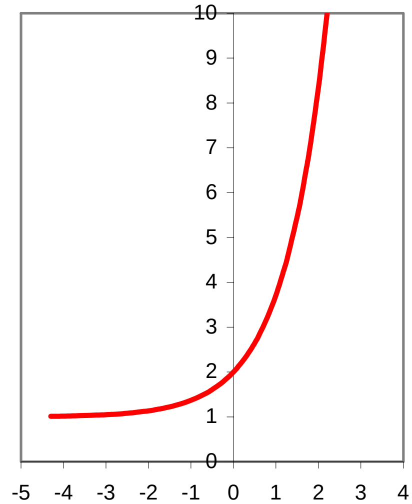
9. The graph of $g(x)=\left(\frac{3}{2}\right)^x+2$ is obtained by shifting $f(x)=\left(\frac{3}{2}\right)^x$ 2 units in y direction.



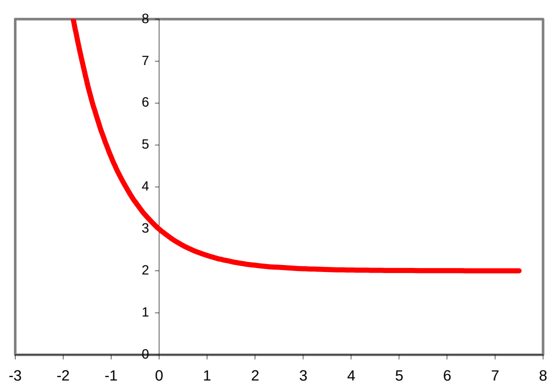
E. The number *e*, Radioactive Decay, and savings accounts

Exercises:

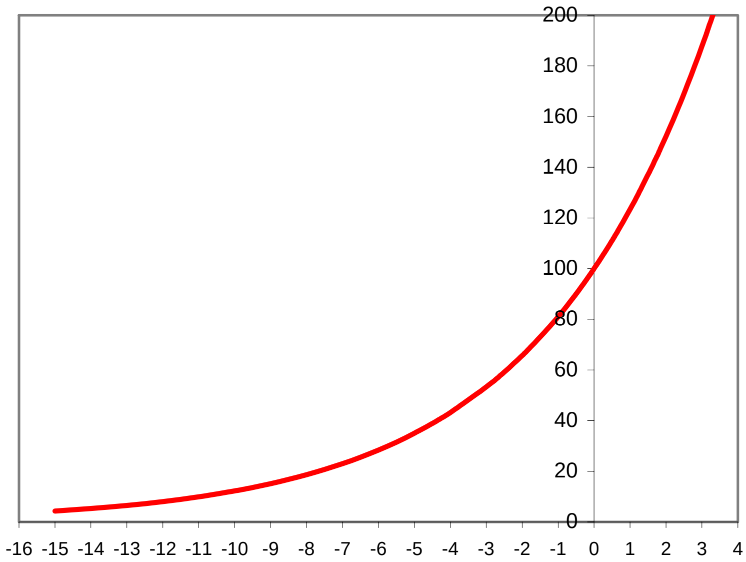
1. To draw the graph of $p(t) = e^t + 1$ we need to shift the graph of $f(t) = e^t$ 1 unit in *y* direction.



2. To draw the graph of $g(t) = e^{-t} + 2$ we need to reflect the graph of $f(t) = e^t$ across the *y* axis and then shift it 2 units in *y* direction.



3. The graph of $f(t) = 100e^{0.21t}$ is given below.



4.

t **(measured in years)**	**Percentage remaining** $e^{-0.015t}$
1	0.98511194 = 98.51%
10	0.860707977 = 86.07%
50	0.472366554 = 47.23%
100	0.223130161 = 22.31%
500	0.000553084 = 0.05%

5.

t **(measured in years)**	**Percentage remaining** $e^{-0.0045t}$
1	0.99551011 = 99.55%
10	0.955997482 = 95.59%
50	0.798516219 = 79.85%
100	0.637628153 = 60.76%
500	0.105399225 = 10.53%

6. We want to estimate the time when the percentage remaining is 50%. If you look at the table in Exercise-4, you'll see that after 50 years 47.23% of the substance is left. So, ½ life of this substance must be smaller than 50.

Percentage remaining is given by $e^{-0.015t}$. Thus, we need to find t such that $e^{-0.015t} = 0.50$. If we take the natural logarithms of both sides, we get $-0.015t = \ln(0.5)$. That is,

$$t = \frac{\ln(0.5)}{-0.015} = 46.2098.$$

Thus, the 1/2 life of this substance is approximately 46.2 years.

7. We know that A_0 dollars will be worth $A(t) = A_0 e^{rt}$ dollars after t years. Here, $A_0 = 1000$, $r = 0.08$ and $t = 10$. Since

$$A(10) = 1000e^{(0.08)(10)} = 1000e^{0.8} = 2225.54,$$

there will be 2225.54 dollars in the account after 10 years.

8. In the formula $A(t) = A_0 e^{rt}$, $A_0 = 100$, $r = 0.07$ and $t = 100$. We have

$$A(100) = 100e^{(0.07)(100)} = 100e^{7} = 109663.32.$$

Thus, there is approximately 109,663.32 dollars in the account.