SEQUENCES

Exercises:

1. $a_n = 10 - 2n$;

$a_1 = 10 - 2(1) = 8$,

$a_2 = 10 - 2(2) = 6$,

$a_3 = 10 - 2(3) = 4$,

$a_4 = 10 - 2(4) = 2$,

$a_5 = 10 - 2(5) = 0$, and

$a_8 = 10 - 2(8) = -6$.

2. $a_n = \dfrac{3}{1-2n}$;

$a_1 = \dfrac{3}{1-2(1)} = \dfrac{3}{-1} = -3$,

$a_2 = \dfrac{3}{1-2(2)} = \dfrac{3}{-3} = -1$,

$a_3 = \dfrac{3}{1-2(3)} = \dfrac{3}{-5} = -\dfrac{3}{5}$,

$a_4 = \dfrac{3}{1-2(4)} = \dfrac{3}{-7} = -\dfrac{3}{7}$,

$a_5 = \dfrac{3}{1-2(5)} = \dfrac{3}{-9} = -\dfrac{1}{3}$, and

$a_8 = \dfrac{3}{1-2(8)} = \dfrac{3}{-15} = -\dfrac{1}{5}$.

3. $a_n = \dfrac{2n-4}{n^2+1}$;

$a_1 = \dfrac{2(1)-4}{(1)^2+1} = \dfrac{-2}{2} = -1$,

$a_2 = \dfrac{2(2)-4}{(2)^2+1} = \dfrac{0}{5} = 0$,

$a_3 = \dfrac{2(3)-4}{(3)^2+1} = \dfrac{2}{10} = \dfrac{1}{5}$,

$a_4 = \dfrac{2(4)-4}{(4)^2+1} = \dfrac{4}{17}$,

$a_5 = \dfrac{2(5)-4}{(5)^2+1} = \dfrac{6}{26} = \dfrac{3}{13}$, and

$a_8 = \dfrac{2(8)-4}{(8)^2+1} = \dfrac{12}{65}$.

4. $a_n = 8 + \frac{1}{n}$;

$a_1 = 8 + \frac{1}{1} = 9$,

$a_2 = 8 + \frac{1}{2} = \frac{17}{2}$,

$a_3 = 8 + \frac{1}{3} = \frac{25}{3}$,

$a_4 = 8 + \frac{1}{4} = \frac{33}{4}$,

$a_5 = 8 + \frac{1}{5} = \frac{41}{5}$, and

$a_8 = 8 + \frac{1}{8} = \frac{65}{8}$.

5. $a_n = 6$;

$a_1 = a_2 = a_3 = a_4 = a_5 = a_8 = 6$.

6. $a_n = 2 + (-1)^n$;

$a_1 = 2 + (-1)^1 = 2 - 1 = 1$,

$a_2 = 2 + (-1)^2 = 2 + 1 = 3$,

$a_3 = 2 + (-1)^3 = 2 - 1 = 1$,

$a_4 = 2 + (-1)^4 = 2 + 1 = 3$,

$a_5 = 1$ and $a_8 = 3$.

7. $a_n = 2 + (0.1)^n$;

$a_1 = 2 + (0.1)^1 = 2.1$,

$a_2 = 2 + (0.1)^2 = 2 + 0.01 = 2.01$,

$a_3 = 2 + (0.1)^3 = 2 + 0.001 = 2.001$,

$a_4 = 2 + (0.1)^4 = 2 + 0.0001 = 2.0001$,

$a_5 = 2 + (0.1)^5 = 2 + 0.00001 = 2.00001$, and

$a_8 = 2 + (0.1)^8 = 2 + 0.00000001 = 2.00000001$.

8. $a_n = (-1)^{n-1} \frac{n+1}{2n}$;

$a_1 = (-1)^{1-1} \frac{1+1}{2(1)} = \frac{2}{2} = 1$,

$a_2 = (-1)^{2-1} \frac{2+1}{2(2)} = -\frac{3}{4}$,

$$a_3 = (-1)^{3-1}\frac{3+1}{2(3)} = \frac{4}{6} = \frac{2}{3},$$

$$a_4 = (-1)^{4-1}\frac{4+1}{2(4)} = -\frac{5}{8},$$

$$a_5 = (-1)^{5-1}\frac{5+1}{2(5)} = \frac{6}{10} = \frac{3}{5}, \text{ and}$$

$$a_8 = (-1)^{8-1}\frac{8+1}{2(8)} = -\frac{9}{16}.$$

9. $a_n = \dfrac{1+(-1)^{n+1}}{2}$;

$$a_1 = \frac{1+(-1)^{1+1}}{2} = \frac{2}{2} = 1,$$

$$a_2 = \frac{1+(-1)^{2+1}}{2} = \frac{1-1}{2} = 0,$$

$a_3 = a_5 = 1$ and $a_4 = a_8 = 0$.

10. $a_n = \dfrac{2^n}{n^2+2}$;

$$a_1 = \frac{2^1}{(1)^2+2} = \frac{2}{3},$$

$$a_2 = \frac{2^2}{(2)^2+2} = \frac{4}{6} = \frac{2}{3},$$

$$a_3 = \frac{2^3}{(3)^2+2} = \frac{8}{11},$$

$$a_4 = \frac{2^4}{(4)^2+2} = \frac{16}{18} = \frac{8}{9},$$

$$a_5 = \frac{2^5}{(5)^2+2} = \frac{32}{27}, \text{ and}$$

$$a_8 = \frac{2^8}{(8)^2+2} = \frac{256}{66} = \frac{128}{33}.$$

11. $a_1 = 1,\ a_{n+1} = \dfrac{a_n+1}{2}$;

$$a_2 = \frac{a_1+1}{2} = \frac{1+1}{2} = 1,$$

$$a_3 = \frac{a_2+1}{2} = \frac{1+1}{2} = 1,$$

$a_4 = a_5 = 1$.

Since all the terms are same, the general formula is $a_n = 1$.

12. $a_1 = 1,\ \ a_{n+1} = \dfrac{1}{n+1}a_n$;

$a_2 = \dfrac{1}{1+1}a_1 = \dfrac{1}{2}(1) = \dfrac{1}{2}$,

$a_3 = \dfrac{1}{2+1}a_2 = \dfrac{1}{3}\left(\dfrac{1}{2}\right) = \dfrac{1}{6}$,

$a_4 = \dfrac{1}{3+1}a_3 = \dfrac{1}{4}\left(\dfrac{1}{6}\right) = \dfrac{1}{24}$,

$a_5 = \dfrac{1}{4+1}a_4 = \dfrac{1}{5}\left(\dfrac{1}{24}\right) = \dfrac{1}{120}$

The general rule is $a_n = \dfrac{1}{n!}$.

13. $a_1 = 2,\ \ a_{n+1} = 2a_n + 2$;

$a_2 = 2a_1 + 2 = 2(2) + 2 = 6$,

$a_3 = 2a_2 + 2 = 2(6) + 2 = 14$,

$a_4 = 2a_3 + 2 = 2(14) + 2 = 30$,

$a_5 = 2a_4 + 2 = 2(30) + 2 = 62$.

14. $a_1 = 1,\ \ a_{n+1} = a_n + 2n + 1$,

$a_2 = a_1 + 2(1) + 1 = 1 + 2 + 1 = 4$,

$a_3 = a_2 + 2(2) + 1 = 4 + 4 + 1 = 9$,

$a_4 = a_3 + 2(3) + 1 = 9 + 6 + 1 = 16$,

$a_5 = a_4 + 2(4) + 1 = 16 + 8 + 1 = 25$.

The general rule is $a_n = n^2$.

15. $a_1 = 1,\ \ a_{n+1} = a_n + a_{n-1} + ... + a_1$;

$a_2 = a_1 = 1$;

$a_3 = a_2 + a_1 = 1 + 1 = 2$,

$a_4 = a_3 + a_2 + a_1 = 2 + 1 + 1 = 4$,

$a_5 = a_4 + a_3 + a_2 + a_1 = 4 + 2 + 1 + 1 = 8$.

16. $a_1 = 1,\ \ a_{n+1} = 2a_n + 1$;

$a_2 = 2a_1 + 1 = 2(1) + 1 = 3$,

$a_3 = 2a_2 + 1 = 2(3) + 1 = 7$,

$a_4 = 2a_3 + 1 = 2(7) + 1 = 15$,

$a_5 = 2a_4 + 1 = 2(15) + 1 = 31$.

17. Since $a_n = \frac{n-1}{n} = 1 - \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, we can conclude that $\lim_{n\to\infty} a_n = 1 - 0 = 1$.

18. In the expression $a_n = \frac{n^2}{n+1}$, the numerator increases much faster than the denominator. Look at some of the terms;

$$a_1 = \frac{1}{2},\ a_2 = \frac{4}{3},\ a_3 = \frac{9}{4},\ a_4 = \frac{16}{5},\ a_5 = \frac{25}{6},\ a_{10} = \frac{100}{11}.$$

So, the terms of this sequence increase a lot as $n \to \infty$ and there is no one number that they get closer. Hence, the sequence $a_n = \frac{n^2}{n+1}$ does not have a limit.

19. We know that $a_n = \frac{n-1}{n^2} = \frac{1}{n} - \frac{1}{n^2}$. Moreover, $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$ as $n \to \infty$. Hence, $a_n \to 0$ as $n \to \infty$.

20. We have; $a_n = \frac{n + (-1)^n}{n} = 1 + \frac{(-1)^n}{n}$. So, we need to know whether $\frac{(-1)^n}{n}$ has a limit or not. Let's look at some of its terms;

$$\frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \ldots, \frac{-1}{999}, \frac{1}{1000}, etc.$$

Although the terms are oscillating between positive and negative numbers, they are getting closer and closer to zero from right and left. Hence, we can conclude that $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

That is, $a_n = \frac{n + (-1)^n}{n} = 1 + \frac{(-1)^n}{n} \to 1$ as $n \to \infty$.

21. For $a_n = \frac{2^n}{4^n + 1}$, the denominator increases much faster than the numerator. Look at the first few terms to see the behavior;

$$a_1 = \frac{2}{5}, a_2 = \frac{4}{17}, a_3 = \frac{8}{65} \approx 0{,}123, a_4 = \frac{16}{256} \approx 0{,}063.$$

So, not only the terms are decreasing they are also getting closer to 0.

Hence, $\lim_{n\to\infty} \left(\frac{2^n}{4^n + 1} \right) = 0$.

22. Let's write down some of the terms of $a_n = \frac{4n}{\sqrt{n^2+1}}$;

$$a_1 = \frac{4}{\sqrt{2}} \approx 2.83, a_2 = \frac{8}{\sqrt{5}} \approx 3.58, a_3 = \frac{12}{\sqrt{10}} \approx 3.79, a_4 = \frac{16}{\sqrt{17}} \approx 3.88, a_5 = \frac{20}{\sqrt{26}} \approx 3.92, a_{100} = \frac{400}{\sqrt{10001}} \approx 3.9998$$

.

Hence, the terms are increasing and getting closer to 4. But, are we sure that the terms do not pass 4? We can not be sure just by checking some of the terms; checking even the 100^{th} term is not enough.

Notice that since $\sqrt{n^2} = n$, $\sqrt{n^2+1} > n$. Hence, $\frac{n}{\sqrt{n^2+1}} < 1$; that is, $a_n = \frac{4n}{\sqrt{n^2+1}} < 4$.

Now we can conclude that $\lim_{n\to\infty}\left(\frac{4n}{\sqrt{n^2+1}}\right) = 4$.

23. $-11, -16, -21, \ldots$ can be the first three terms of an arithmetic sequence since there is a constant difference between the terms; the terms are **decreasing** by 5. Therefore, $d = -5$ ($a_2 - a_1 = -16 - (-11) = -5$).

We know the first term; $a = -11$. Then,

$$a_n = a + (n-1)d = -11 + (n-1)(-5) = -11 - 5n + 5 = -6 - 5n.$$

24. Let's check the difference and the ratio between the terms of $1, 4, 9, \ldots$.

$a_2 - a_1 = 4 - 1 = 3$ but $a_3 - a_2 = 9 - 4 = 5$; the difference is not constant.

$\frac{a_2}{a_1} = \frac{4}{1} = 4$ but $\frac{a_3}{a_2} = \frac{9}{4}$; the ratio is not constant.

Hence, these three terms can not be the terms of an arithmetic or a geometric sequence.

25. The sequence $2, -4, 8, \ldots$ is a geometric sequence since there is a constant ratio;

$$\frac{a_2}{a_1} = \frac{-4}{2} = -2 \text{ and } \frac{a_3}{a_2} = \frac{8}{-4} = -2.$$

Hence, $r = -2$ and $a = 2$.

The general term is $a_n = a \cdot r^{n-1} = 2(-2)^{n-1}$.

26. $7, 6.5, 6, \ldots$. can be an arithmetic sequence since $6.5 - 7 = 6 - 6.5 = -0.5$. The common difference is $d = -0.5$.

The general term is;

$$a_n = a + (n-1)d = 7 + (n-1)(-0.5) = 7 - 0.5n + 0.5 = -0.5n + 7.5.$$

27. For the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$, it is obvious that the difference between the consecutive terms is not constant. Let's check the ratio;

$$\frac{a_2}{a_1} = \frac{1/6}{1/2} = \frac{1}{3} \text{ and } \frac{a_3}{a_2} = \frac{1/18}{1/6} = \frac{1}{3}.$$

Then, the sequence can be a geometric sequence with the common ratio $r = \frac{1}{3}$ and $a = \frac{1}{2}$.
Hence, the general term is:

$$a_n = a \cdot r^{n-1} = \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}.$$

28. $3,3,3,...$ is both an arithmetic and a geometric sequence since there is a constant difference and a constant ratio between the consecutive terms;

$$3-3=3-3=0 \text{ and } \frac{3}{3}=\frac{3}{3}=1.$$

The general rule is $a_n = 3(1)^{n-1} = 3+(n-1)0 = 3$.

29. It is clear that the difference is not constant between the consecutive terms of $\frac{1}{2},\frac{1}{4},\frac{1}{9},....$ Let's check the ratio between them.

$$\frac{a_2}{a_1} = \frac{1/4}{1/2} = \frac{1}{2} \text{ and } \frac{a_3}{a_2} = \frac{1/9}{1/4} = \frac{4}{9}.$$

So, there is no constant ratio. This sequence is neither arithmetic nor geometric.

30. The sequence $7,14,28,...$ is a geometric sequence since $\frac{a_2}{a_1} = \frac{14}{7} = 2$ and $\frac{a_3}{a_2} = \frac{28}{14} = 2$.
Then, the common ratio is $r=2$ and $a=7$. The general term is:

$$a_n = a \cdot r^{n-1} = 7(2)^{n-1}.$$

31. For the arithmetic sequence $2,6,10,14,...$, the common difference is $d = 6-2 = 10-6 = 4$ and $a=2$.
The 10$^{\text{th}}$ term is: $a_{10} = 2+(10-1)4 = 2+36 = 38$
The n^{th} term is: $a_n = a+(n-1)d = 2+(n-1)(4) = 2+4n-4 = 4n-2$.
Finding the 10$^{\text{th}}$ term is easier after finding the n^{th} term; $a_{10} = 4(10)-2 = 38$.

32. The common difference for the arithmetic sequence $11,9,7,...$ is -2; $d = 9-11 = -2$.
Since the first term is 11, the n^{th} term is;

$$a_n = a+(n-1)d = 11+(n-1)(-2) = 11-2n+2 = -2n+13.$$

And the 10$^{\text{th}}$ term is: $a_{10} = -2(10)+13 = -20+13 = -7$.

33. For the arithmetic sequence $3, 2.7, 2.4, 2.1,...$, $d=-0.3$ and $a=3$. Hence, the n^{th} term is:

$$a_n = a+(n-1)d = 3+(n-1)(-0.3) = 3-0.3n+0.3 = -0.3n+3.3.$$

And the 10th term is: $a_{10} = 3 + (10-1)(-0.3) = 3 - 2.7 = 0.3$.

34. The 10th term is $a_{10} = 2.5$ and $a_n = 7.5 - \frac{n}{2}$.

35. Since $a = 7$ and $d = 2$, $a_5 = a + 4d = 7 + 4(2) = 15$ and $S_{10} = \frac{10}{2}[2(7) + 9(2)] = 5(32) = 160$.

36. $a = 2$ and $d = -3$; $a_{10} = a + 9d = 2 + 9(-3) = -25$ and
$S_{20} = \frac{20}{2}[2(2) + 19(-3)] = 10(-53) = -530$.

37. $a = 5$ and $d = 0.1$; $a_{10} = a + 9d = 5 + 9(0.1) = 5.9$ and
$S_{20} = \frac{20}{2}[2(5) + 19(0.1)] = 10(11.9) = 119$.

38. $a = \frac{7}{3}$ and $d = -\frac{2}{3}$; $a_{16} = a + 15d = \frac{7}{3} + 15(-\frac{2}{3}) = -\frac{23}{3}$ and
$S_{16} = \frac{16}{2}\left[2\left(\frac{7}{3}\right) + 15\left(-\frac{2}{3}\right)\right] = 8\left(-\frac{16}{3}\right) = -\frac{128}{3}$.

39. For the geometric sequence $8, 4, 2, 1, \ldots$, the common ratio is: $r = \frac{4}{8} = \frac{1}{2}$ and $a = 8$
(Notice that we can use $r = \frac{a_2}{a_1}$ since we already know that the sequence is a geometric sequence.).
The nth term is: $a_n = a \cdot r^{n-1} = 8\left(\frac{1}{2}\right)^{n-1}$.
The 10th term is: $a_{10} = a \cdot r^{10-1} = 8\left(\frac{1}{2}\right)^9 = \left(\frac{1}{2}\right)^6$.

40. $2, 6, 18, \ldots.$ is a geometric sequence with $r = \frac{6}{2} = 3$ and $a = 2$.
The nth term is: $a_n = a \cdot r^{n-1} = 2(3)^{n-1}$.
The 10th term is: $a_{10} = a \cdot r^{10-1} = 2(3)^9$.

41. The common ratio of $4, 1.2, 0.36, \ldots$ is $r = \frac{1.2}{4} = 0.3$ and $a = 4$.
The nth term is: $a_n = a \cdot r^{n-1} = 4(0.3)^{n-1}$.
The 10th term is: $a_{10} = a \cdot r^{10-1} = 4(0.3)^9$.

42. For the geometric sequence $4,-6,9,-13.5,\ldots$, $r=\frac{-6}{4}=-\frac{3}{2}$ and $a=4$.

The n^{th} term is: $a_n=a\cdot r^{n-1}=4\left(-\frac{3}{2}\right)^{n-1}$.

The 10^{th} term is: $a_{10}=a\cdot r^{10-1}=4\left(-\frac{3}{2}\right)^{9}=-\frac{3^9}{2^7}$.

43. $a=3,\ \ r=-2$; $a_6=a\cdot r^{6-1}=3(-2)^5=3(-32)=-96$ and

$S_6=\frac{a-ar^6}{1-r}=\frac{3-3(-2)^6}{1-(-2)}=\frac{3-3(64)}{3}=-63$.

44. $a=2,\ \ r=\frac{1}{3}$; $a_4=a\cdot r^{4-1}=2\left(\frac{1}{3}\right)^3=\frac{2}{27}$ and $S_4=\frac{2-2(1/3)^6}{1-(1/3)}=3-\frac{1}{3^5}=\frac{3^6-1}{3^5}$.

45. $a=1,\ \ r=\frac{3}{2}$; $a_4=a\cdot r^{4-1}=1\left(\frac{3}{2}\right)^3=\frac{27}{8}$ and $S_4=\frac{1-(3/2)^6}{1-(3/2)}=-2\left[1-(3/2)^6\right]=-2+\frac{3^6}{2^5}=$.

46. $a=50,\ \ r=\frac{1}{2}$; $a_{10}=a\cdot r^{10-1}=50\left(\frac{1}{2}\right)^9$ and

$S_{10}=\frac{50-50(1/2)^{10}}{1-(1/2)}=2\left[50-50(1/2)^{10}\right]=100-100\left(\frac{1}{2}\right)^{10}=100-\frac{25}{2^8}$.

47. The series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ is a geometric series where $a=1$ and

$r=\frac{-1/2}{1}=\frac{1/4}{-1/2}=-\frac{1}{2}$. Since $|r|=\frac{1}{2}<1$, this series has a finite sum and the sum is

$S=\frac{a}{1-r}=\frac{1}{1-(-1/2)}=\frac{2}{3}$.

48. $50+25+12.5+\ldots.$ is also a geometric series with $a=50$ and $r=\frac{25}{50}=\frac{12.5}{25}=\frac{1}{2}$.

Moreover, $|r|=\frac{1}{2}<1$. Thus, the series has a finite sum which is $S=\frac{a}{1-r}=\frac{50}{1-1/2}=100$.

49. $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots$ is a geometric series with $a=2$ and $r=\frac{2/3}{2}=\frac{2/9}{2/3}=\frac{1}{3}$. Since

$|r|=\frac{1}{3}<1$. The series has a finite sum and the sum is $S=\frac{a}{1-r}=\frac{2}{1-1/3}=3$.

50. The series $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+...$ is a geometric series where $a=1$ and $r=\frac{3/2}{1}=\frac{9/4}{3/2}=\frac{27/8}{9/4}=\frac{3}{2}$. However, $|r|=\frac{3}{2}>1$. This means that this series does not have a finite sum.