QUADRATIC FUNCTIONS

A.

Exercises:

1. $(-2x+1)(3x-5) = -2x(3x-5) + 1(3x-5) = -6x^2 + 10x + 3x - 5 = -6x^2 + 13x - 5$.

2. $(x-7)(5x-6) = x(5x-6) - 7(5x-6) = 5x^2 - 6x - 35x + 42 = 5x^2 - 41x + 42$.

3.

$\times$	$3x$	-5
$-2x$	$-6x^2$	$10x$
1	$3x$	-5
Sum:	$-6x^2 + 10x + 3x - 5$	
$(-2x+1)(3x-5) =$	$-6x^2 + 13x - 5$	

4.

$\times$	$5x$	-6
x	$5x^2$	$-6x$
-7	$-35x$	42
Sum:	$5x^2 - 6x - 35x + 42$	
$(x-7)(5x-6) =$	$5x^2 - 41x + 42$	

5. $(-2x+1)(3x-5) = -6x^2 + 10x + 3x - 5 = -6x^2 + 13x - 5$.

6. $(x-7)(5x-6) = 5x^2 - 6x - 35x + 42 = 5x^2 - 41x + 42$.

7. $\left(-2x + \frac{1}{2}\right)(5x-2) = -2x(5x-2) + \frac{1}{2}(5x-2) = -10x^2 + 4x + \frac{5}{2}x - 1 = -10x^2 + \frac{13}{2}x - 1$.

8. We'll expand it by using the distributive property;
$(10x+1)(10x-1) = 10x(10x-1) + 1(10x-1) = 100x^2 - 10x + 10x - 1 = 100x^2 - 1$.

9. Let's use the FOIL method;
$(ax+b)(ax-b) = a^2x^2 - axb + bax - b^2 = a^2x^2 - b^2$.

10. To factor x^2+3x-4, we need to find *p* and *q* so that $pq=-4$ and $p+q=3$.If you try some possibilities for $pq=-4$, you'll see that when $p=4$ and $q=-1$ both conditions are satisfied.
Hence, $x^2+3x-4=(x+4)(x-1)$.

11. To factor the expression $3x^2+4x+1$, we need to find *r*, *p*, *s* and *q* so that $3x^2+4x+1=(rx+p)(sx+q)$. That is, $rs=3$, $ps+rq=4$ and $pq=1$.

Let's try the easiest case; $p=q=1$ and $r=3,\ s=1$. Two of the conditions are satisfied ($rs=3$ and $pq=1$), but we need to check the third:
$ps+rq=(1)(1)+(3)(1)=4$.
So, these are the numbers we're looking for.
Hence, $3x^2+4x+1=(3x+1)(x+1)$.

12. x^2-16 is an expression which is suitable to use the difference of two squares. Notice that $4^2=16$. Hence, $x^2-16=(x-4)(x+4)$.

13. x^2-100 is also suitable for the difference of two squares. Since $10^2=100$, we get:

$$x^2-100=(x-10)(x+10).$$

14. Firstly, notice that we can not use the difference of two squares here (basically since there is no "difference" here, this is a sum).
To factor x^2+100, we need *p* and *q* so that $pq=100$ and $p+q=0$. There are lots of couples of numbers satisfying the first condition, but it is important to remember that to have $p+q=0$, *p* and *q* must be of opposite sign (we know that they are not 0 since their product is not). However, if *p* and *q* have opposite signs then their product should be negative. For example, $p=10,\ q=-10 \Rightarrow pq=-100$. Hence, it is impossible to find such two numbers. The expression x^2+100 is not a product of linear factors with integer coefficients.

15. To factor x^2+x+6, we need *p* and *q* so that $pq=6$ and $p+q=1$.

p and *q* so that $pq=6$	$p+q$
1 and 6	7
2 and 3	5

Hence, as you can observe from the table, x^2+x+6 does not have linear factors with integer coefficients.

16. In Exercise-9, we expanded the expression $(ax+b)(ax-b)$ and we got $(ax+b)(ax-b)=a^2x^2-b^2$. So, the factors of $a^2x^2-b^2$ are $(ax+b)$ and $(ax-b)$;

$$a^2x^2-b^2=(ax+b)(ax-b).$$

17. We know from Exercise-10 that $x^2+3x-4=(x+4)(x-1)$. So, $x^2+3x-4=0$ if and only if $x+4=0$ or $x-1=0$.
Hence, the solutions of the equation $x^2+3x-4=0$ are $x=-4$ and $x=1$.

18. We know that $3x^2+4x+1=(3x+1)(x+1)$ (See Exercise-11). Hence, $3x^2+4x+1=0$ if and only if $3x+1=0$ or $x+1=0$.
That is, the solutions to $3x^2+4x+1=0$ are $x=-\frac{1}{3}$ and $x=-1$.

19.We have $x^2-16=(x-4)(x+4)$, so $x^2-16=0$ if and only if $x-4=0$ or $x+4=0$.
Thus, the solutions of the expression $x^2-16=0$ are $x=4$ and $x=-4$.

20. Since $x^2-100=(x-10)(x+10)$, $x^2-100=0$ if and only if $x-10=0$ or $x+10=0$.
Hence, the solutions to $x^2-100=0$ are $x=10$ and $x=-10$.

21. As we saw before, $a^2x^2-b^2=(ax+b)(ax-b)$. That is, $a^2x^2-b^2=0$ if and only if $ax+b=0$ or $ax-b=0$.
$ax+b=0 \Rightarrow ax=-b \Rightarrow x=-\frac{b}{a}$ and $ax-b=0 \Rightarrow ax=b \Rightarrow x=\frac{b}{a}$.
Hence, the solutions to the equation $a^2x^2-b^2=0$ are $x=-\frac{b}{a}$ and $x=\frac{b}{a}$.

B. Graphs of Quadratic Functions

Exercises:

1. The quadratic formula tells us that $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. For $x^2+x-4=0$, the solutions are:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-4)}}{2} = \frac{-1 \pm \sqrt{17}}{2}.$$

2. The solutions of the equation $2x^2 + x - 7 = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(2)(-7)}}{2(2)} = \frac{-1 \pm \sqrt{57}}{4}.$$

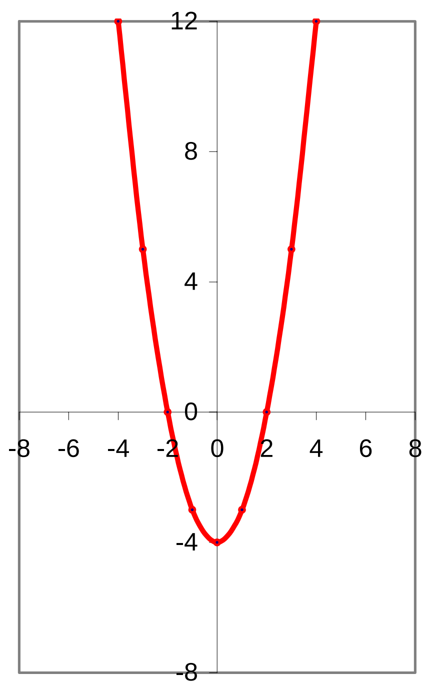
3. The equation $x^2 - 2x = 5$ can be written as $x^2 - 2x - 5 = 0$. Now, we can use the quadratic formula. The solutions are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(1)(-5)}}{2(1)} = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}.$$

4. $2x^2 = 3x + 11 \Rightarrow 2x^2 - 3x - 11 = 0$.
The solutions are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 - 4(2)(-11)}}{2(2)} = \frac{3 \pm \sqrt{97}}{4}.$$

5. The graph of $f(x) = x^2 - 4$ is obtained by shifting the graph of $y = x^2$ -4 units in the y direction.



We can easily see from the graph that the *x* intercepts are $x = \pm 2$, the y intercept is $y = -4$, and the vertex is $(0,-4)$.

We can also use the following reasoning to find the vertex and the *x* intercepts (which is very useful when we do not know the graph of the function):

- The *x* coordinate of the vertex of $f(x) = x^2 - 4$ is given by $-\frac{b}{2a} = -\frac{0}{2} = 0$. Since $f(0) = -4$, the vertex is $(0,-4)$.
- The *x* intercepts are the solutions of $x^2 - 4 = 0$. By factoring or by using the quadratic formula, we can easily get that $x = \pm 2$ are the solutions and hence the *x* intercepts.
- The *y* intercept is $f(0) = -4$.

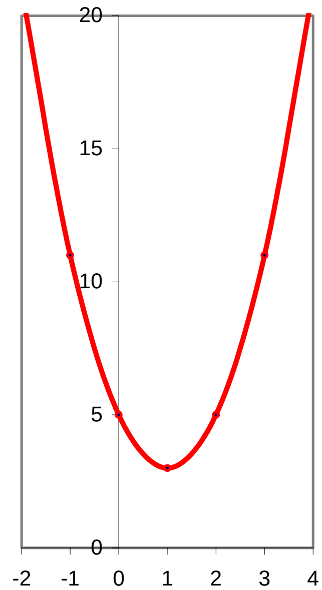
6. It is not easy to draw the graph of $f(x) = 2x^2 - 4x + 5$ by using $y = x^2$. So, this time let's firstly find the vertex and *x* intercepts and then draw the graph accordingly.

The *x* coordinate of the vertex of $f(x) = 2x^2 - 4x + 5$ is given by $-\frac{b}{2a} = -\frac{-4}{4} = 1$. Since $f(1) = 2(1)^2 - 4(1) + 5 = 3$, the vertex is $(1,3)$.

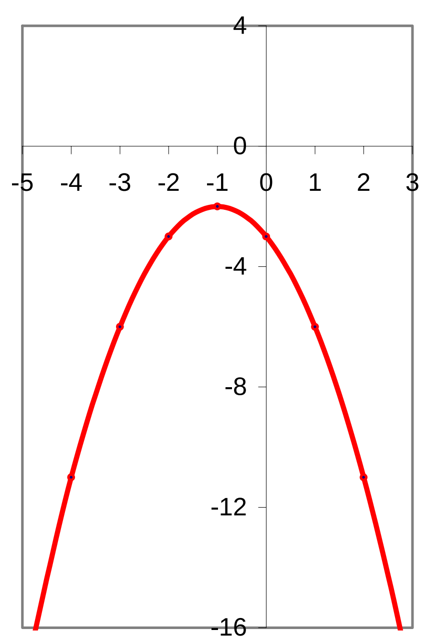
The *x* intercepts are the solutions of $2x^2 - 4x + 5 = 0$. However, for this equation the discriminant is $b^2 - 4ac = 16 - 4(2)(5) = -24 < 0$ which means that there are no solutions. Hence, there are no *x* intercepts.

Since there are no *x* intercepts, we need another point on the graph to be able to draw it. We have $f(0) = 5$, so the *y* intercept is $(0,5)$.

Now, since $a > 0$, draw a parabola with vertex $(1,3)$ that turns up and passes through $(0,5)$. Remember that the arms of the parabola are symmetric.

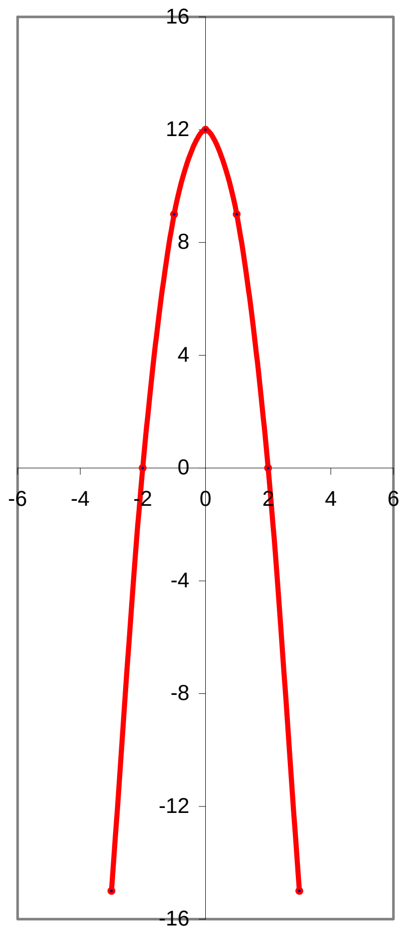


7. $f(x) = -x^2 - 2x - 3 = -(x^2 + 2x + 3) = -(x+1)^2 - 2$. Hence, the graph of $f(x) = -x^2 - 2x - 3 = -(x+1)^2 - 2$ can be obtained by shifting the graph of $f(x) = x^2$ firstly -1 units in the x direction and then -2 units in the y direction. The vertex is : $(-1, -2)$ and the parabola is turning down. And this shows that there are no x intercepts. The y intercept is $y = f(0) = -3$.



8. To draw the graph of $f(x) = -3x^2 + 12$:

- Scale $f(x) = -x^2$ by 3 to get $f(x) = -3x^2$
- Shift $f(x) = -3x^2$ 12 units in y direction.



As it is seen from the graph, the vertex is $(0,12)$ and the x intercepts are $x = \pm 2$. The y intercept is $y = 12$.

C. Applications

1. Their fixed costs are \$10,000 per month and each pair costs \$20, so the monthly cost is:

$$C(x) = 10,000 + 20x.$$

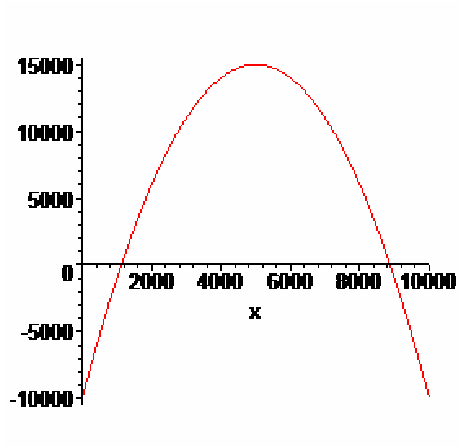
They can sell x pairs of sandals each month at a price of $40 - \frac{x}{100}$, then their monthly revenue is:

$$R(x) = x\left(30 - \frac{x}{1000}\right) = 30x - \frac{x^2}{1000}.$$

Thus, the profit function is:

$$P(x) = R(x) - C(x) = 30x - \frac{x^2}{1000} - (10,000 + 20x) = -\frac{x^2}{1000} + 10x - 10,000.$$

The graph of the profit function is:



So the break even occurs when approximately 1,010 or 99,030 pairs of sandals are produced. Profit is maximized by selling 50,020 pairs of sandals.

2. We know that $s(t) = -4.9t^2 + v_0 t + s_0$ gives the position of a particle at a time $t \geq 0$. Here $v_0 = 25$ and $s_0 = 25$, then;

$$s(t) = -4.9t^2 + 25t + 25.$$

The particle strikes the ground at the time t where $s(t) = -4.9t^2 + 25t + 25 = 0$. This occurs when $t \approx 5.95$, that is the particle stays in the air for approximately 5.95 seconds.

The particle is at the maximum height when $t = -\frac{25}{2(-4.9)} \approx 2.55$.

And the maximum height that the particle reaches is : $s(2.55) = 56.89$ meters.