

Linear Functions

A. Definition and Examples

A function f is linear if it can be expressed in the form

$$f(x) = mx + b$$

where m and b are constants and x is an arbitrary member of the domain of f . Often the relationship between two variables x and y is a linear function expressed as an equation $y = mx + b$.

Examples:

- (1) Let F denote the Fahrenheit temperature and C the Celsius temperature of an object. F and C are related by

$$F = \frac{9}{5}C + 32$$

Thus, F is a linear function of C with $m = \frac{9}{5}$ and $b = 32$.

- (2) Let v be the downward velocity of a body falling freely near the surface of the Moon under the influence of gravity and let t be the time it has fallen. Let v_0 be the velocity imparted to it initially. Then

$$v = gt + v_0.$$

Thus, v is a linear function of t with $m = g$, the acceleration due to gravity near the surface of the Moon, and $b = v_0$.

- (3) Suppose that you put P dollars into a savings account that draws simple interest at a rate of r . After time t , the value of the savings account will be

$$V(t) = P(1 + rt).$$

V is a linear function of t . In this example, it is not quite so obvious what the constants m and b are. We leave it as an exercise to identify them.

- (4) A school collects \$50,000 each year from the government for a school lunch program, plus an additional \$500 for each student classified as economically disadvantaged. The amount of money collected is a linear function of the number of economically disadvantaged students enrolled in the school.

Exercise 1: Make up symbols for the variables in Example 4 and write the expression for the linear function relating them.

Exercise 2: Which of the following functions are linear?

- (a) $f(x) = -2x - 5$ (b) $g(t) = 0.25t^2 + 1$ (c) $h(s) = 1000(1-2s)$
(d) $k(r) = 2/(r-10)$

B. Change and Rate of Change

Let f be a function whose domain is a set of numbers. If x_1 and x_2 are distinct members of the domain, the change (or increment) in f from x_1 to x_2 is the number $f(x_2) - f(x_1)$.

The rate of change from x_1 to x_2 is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Sometimes the change in the value of the independent variable x is denoted by Δx and the corresponding change in the value of the function is denoted by Δf . Using this shorthand notation, the rate of change is $\frac{\Delta f}{\Delta x}$.

Exercise 3: For real numbers x , let $f(x) = x^2$. Find the change in f from $x_1 = 1$ to $x_2 = 4$. Find the rate of change of f over this interval. Find the rate of change of f over the interval from 0 to 3.

Exercise 4: Find a general formula for the rate of change of f in Exercise 3 over the interval from x_1 to x_2 for any x_1 and x_2 .

From the preceding exercise, you can see that some functions have different rates of change over different intervals. The most important property of linear functions is that a linear function has the same rate of change over all intervals. Indeed, if $f(x) = mx + b$ then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

Thus, the parameter m in the general formula for a linear function has an important meaning. It is the constant rate of change of the function. The converse is also true. If a function has a constant rate of change, then it is linear. The parameter b is the value of the function when $x = 0$.

If you know that a function is linear and also know its change over a particular interval, then you can easily find its change over any other interval. The following example illustrates this.

Example 5: The amount of money a school gets for its free lunch program is a linear function of the number of economically disadvantaged students enrolled at the school. Last year, the school had 50 fewer e.d. students than in the previous year and its funding was reduced by \$35000. This year, there will be 30 more e.d. students than there were last year. How much will funding change from last year to this year?

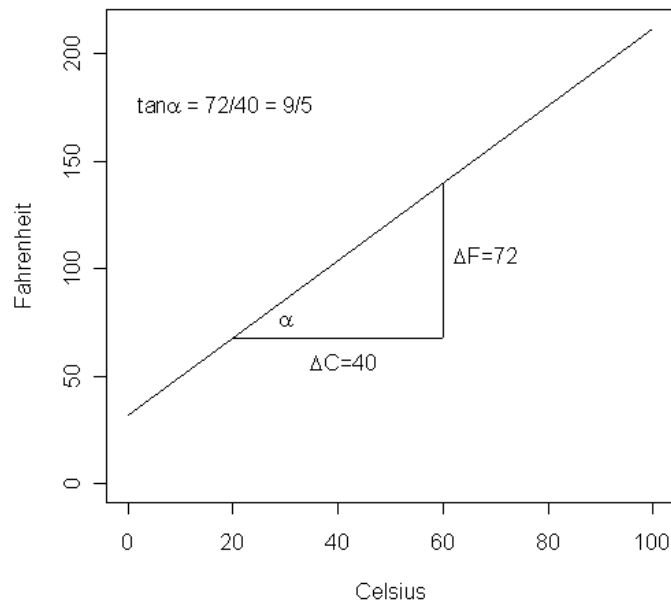
Solution: Between the year before last and last year, the change in the number of e.d. students was $\Delta x = -50$ and the corresponding change in the value of funding was $\Delta f = -35000$. Therefore, the rate of change of the function is $m = \frac{-35000}{-50} = 700$. That is, the school gets 700 additional dollars for each additional e.d. student it enrolls. From last year to this year, the value of Δx is 30. Since $\frac{\Delta f}{\Delta x} = m = 700$, $\Delta f = m\Delta x = 700(30) = \$21,000$. Therefore, the school's funding will be increased by \$21,000 this year.

Exercise 5: Suppose that between year before last and last year the school lost 20 e.d. students and \$24,000 in funding. If there will be an increase of 30 e.d. students this year, by how much will the school's funding change?

C. Graphs of Linear Functions – Straight lines.

If variables x and y are related by a linear function $y = mx + b$ and the graph of this equation is plotted on a rectangular coordinate system, the result will be a straight line. If the variable x is by its nature restricted to some subset of the real number system, the domain of the function is that subset, and the graph of the function might be only a part of a straight line. For simplicity we will ignore that possibility and consider the domain of a linear function to be the entire real number system.

The equation $y = mx + b$ is the *slope-intercept* form of the equation of a straight line. Any non-vertical line in the Cartesian plane has an equation of this form. That is, every non-vertical line is the graph of a linear function. The rate of change m is the slope of the line, or the tangent of the angle from the x -axis to the line. The parameter b is the y -intercept, i.e. the height of the graph where it crosses the vertical line $x = 0$. This is illustrated in the figure below.



Some special cases deserve mention. If the rate of change of a linear function is zero, the values of the function never change. That is, the function is constant. Moreover, the slope of its graph is zero, so the graph is a horizontal line. If the rate of change is positive, the function is increasing and the graph slopes upward to the right. If the rate of change is negative, the function is decreasing and the graph slopes downward to the right. If two straight lines have the same slope, they make the same angle with the positive x-axis and are parallel.

Exercise 6: Sketch the graphs of the following linear functions.

(1) $f(x) = -1 + 2x$

(2) $g(t) = -2t + 5$

(3) $C(F) = \frac{5}{9}(F - 32)$

D. Other Forms of the Equation of a Line

Sometimes it is more convenient to write the equation of a straight line in some form other than the slope-intercept form. In other words, we may want to write the formula for a linear function in some way other than $f(x) = mx + b$. Example 3 above is such a case.

The Point-Slope Form: Suppose it is known that a line passes through the point with coordinates (x_0, y_0) and that it has slope m . Then the equation of the line is

$$y = y_0 + m(x - x_0).$$

Translating to the language of linear functions, if it is known that $f(x_0) = y_0$ and the rate of change is m , then the function is given by the formula $f(x) = y_0 + m(x - x_0)$. If the point-slope form of the equation of a line is $y = y_0 + m(x - x_0)$, then in slope-intercept form $y = mx + b$ the intercept parameter b is equal to $b = y_0 - mx_0$. This is easily seen by remembering that if the line is the graph of f , then $b = f(0)$.

Exercise 7: Suppose that a line passes through the point $(-1,4)$ and has slope -2 . Find its equation in slope-intercept form.

Exercise 8: The equation of a line is $y = -4x + 2$. Find its equation in point-slope form where $x_0 = 3$.

Exercise 9: A line passes through the points $(-1,1)$ and $(4, 0)$. What is its equation?

Example 6: A school's funding for free and reduced lunches increases by \$700 for each additional economically disadvantaged student enrolled. Last year, 250 e.d. students were enrolled and the school collected \$250,000. What is the amount collected if n economically disadvantaged students are enrolled?

Solution: If $f(n)$ denotes the funding for n students, then we know that f is a linear function with rate of change $m = 700$ and that $f(250) = 250,000$. Therefore, a formula for f is $f(n) = 250,000 + 700(n - 250)$. After simplifying, we get $f(n) = 75,000 + 700n$.

The Symmetric Form: The point-slope and slope-intercept forms of the equation of a straight line are perhaps the most convenient forms to use when the line is a graph of a linear function. Remember, however, that a vertical line in the xy plane is not the graph of a function. The symmetric form of the equation of a line includes vertical as well as non-vertical lines. It is

$$cx + dy = e$$

where c , d and e are constants and at least one of c and d is different from zero. Notice that the constants c , d and e are not unique. For example, we could multiply each one by 2 and although the equation would be formally different, it would still represent the same line. Thus, $-x + 3y = 8$ and $2x - 6y = -16$ are both equations in symmetric form of the same straight line.

If the equation of a line is given in slope-intercept form $y = mx + b$, then one way of writing the equation in symmetric form is $-mx + y = b$. Suppose on the other hand that the equation is given in symmetric form $cx + dy = e$ and that $d \neq 0$. By rearranging

terms, we get $y = (-c/d)x + e/d$, which is slope-intercept form with slope $m = -c/d$ and intercept $b = e/d$.

Exercise 10: The graph of a linear function is the line whose equation is $2x - 5y = 8$. What is the rate of change of f ? What are $f(0)$ and $f(-2)$?

Exercise 11: What choices of the constants c , d and e would give the equation of a horizontal line? A vertical line? A line with slope 1? A line with slope -1 ?

E. Systems of Linear Equations

Example 7: Suppose that you put \$2,000 into a savings account drawing simple interest of 3% per year. At the same time, your partner invests \$1,000 in bonds that yield 8% per year. At what time in the future will the two investments have the same value?

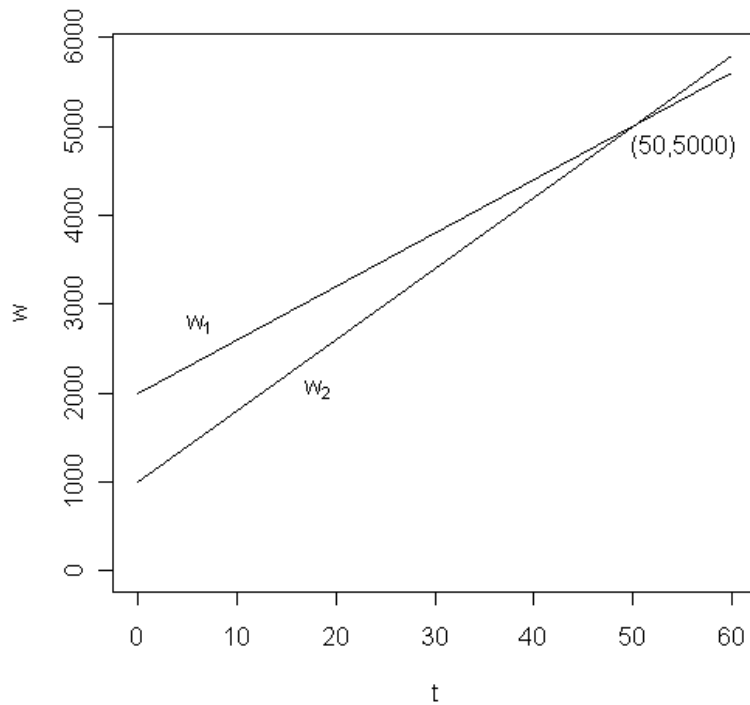
Solution: Your worth and your partner's worth at time t years in the future are both linear functions of t . Yours is

$$w_1(t) = 2000 + 60t$$

and your partner's is

$$w_2(t) = 1000 + 80t$$

Let us graph both functions on the same coordinate system, using the horizontal axis as the t -axis and the vertical axis as the w -axis.



If you do a careful job of graphing, you can confirm that the two straight lines intersect when $t = 50$. Thus you and your partner will have the same worth in 50 years, at which time you are both worth \$5,000. After that your partner will be worth more than you are if you are both still alive.

The equations for the straight lines in the preceding figure may be written in symmetric form as follows.

$$80t - w = -1000$$

$$60t - w = -2000$$

This is called a system of two linear equations in two unknowns, t and w . A solution of the system is a pair (t, w) of numbers that satisfy both equations. In this case, there was only one solution, namely $(50, 5000)$. In general, a system of two linear equations is of the form

$$a_{1,1}x + a_{1,2}y = c_1$$

$$a_{2,1}x + a_{2,2}y = c_2$$

where the a 's and c 's are all constants and x and y are the unknowns. A solution of the system is a pair (x, y) of numbers that satisfy both equations simultaneously. Both equations are equations in symmetric form of straight lines. Therefore, (x, y) is a solution

of the system if and only if the point with coordinates (x, y) lies on both lines, i.e., is a point of intersection of the two lines.

Given two equations for lines in the coordinate plane, there are three possibilities:

- (a) The lines are not parallel and intersect in one and only one point. That is, there is one and only one solution of the system.
- (b) The lines are distinct but parallel and do not intersect. There are no solutions.
- (c) The equations represent the same straight line. There are infinitely many solutions, one for each point on the line.

It is easy to determine whether or not case (a) holds, that is, whether or not the system has a unique solution. The lines are parallel or coincide if and only if they have the same slopes (or they are both vertical). Assuming they are not vertical, case (a) fails to hold if and only if $-\frac{a_{1,1}}{a_{1,2}} = -\frac{a_{2,1}}{a_{2,2}}$. To express it differently, the system does not have a unique solution if $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 0$ and it does have a unique solution if $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$. If the lines are both vertical, then $a_{1,2} = a_{2,2} = 0$ and it is still true that $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 0$.

The coefficients of x and y in the two equations of the system are sometimes arranged in a square array called the coefficient matrix.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

The number $a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ is called the determinant of the coefficient matrix, or the

determinant of the system. It is denoted by $\det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ or by $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$. By the

remarks above, the system has a unique solution if and only if its determinant is different from zero. Furthermore, if the determinant is different from zero, the system will have a unique solution for any choices of the constant terms c_1 and c_2 . There are several methods of finding the solution. One of them is Cramer's Rule:

$$x = \frac{\begin{vmatrix} c_1 & a_{1,2} \\ c_2 & a_{2,2} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_{1,1} & c_1 \\ a_{2,1} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}.$$

Cramer's rule is not necessarily the best way to find the solution of a system of linear equations. In particular, it cannot be used when there are infinitely many solutions. For systems of two equations in two unknowns simple substitution is effective. We illustrate it along with Cramer's rule in the following example.

Example 8: Solve the system

$$\begin{aligned} 2x - 3y &= 0 \\ x + 2y &= 1 \end{aligned}$$

by two methods.

Solution: First we use the method of substitution. From the second equation we have $x = 1 - 2y$. Substituting this expression for x in the first equation gives $2(1 - 2y) - 3y = 0$, or $2 - 7y = 0$. Thus, $y = \frac{2}{7}$ and $x = 1 - 2(\frac{2}{7}) = \frac{3}{7}$. The solution is $(\frac{3}{7}, \frac{2}{7})$. To use Cramer's rule we first find the determinant of the coefficient matrix. It is $\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = (2)(2) - (-3)(1) = 7$. By Cramer's rule,

$$x = \frac{\begin{vmatrix} 0 & -3 \\ 1 & 2 \end{vmatrix}}{7} = \frac{(0)(2) - (-3)(1)}{7} = \frac{3}{7} \text{ and } y = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}}{7} = \frac{2}{7}.$$

Exercise 12: For each of the following systems of linear equations, determine whether the system has a unique solution, no solutions, or infinitely many solutions. If there is a unique solution, find it by two different methods.

$$\begin{array}{ll} \text{(a)} & \begin{cases} 2x - 2y = 1 \\ -x + 4y = 0 \end{cases} \\ \text{(b)} & \begin{cases} -r - 3s = 1 \\ -r + 2s = 4 \end{cases} \end{array}$$

Finally, we remark that there are systems of linear equations with more than two equations and/or more than two unknowns. Such systems are best solved by using some form of technology. Their study is beyond our present scope.

F. Inverses of Linear Functions

Let x and y be variables related by a linear function: $y = f(x) = mx + b$. Normally, this relationship is applied by evaluating the function, that is, by using the formula of the function to calculate the value of y for a given value of x . Sometimes however, the value of y is given and the goal is to solve for the value of x that y comes from. If $m \neq 0$, there is one and only one such value of x for any given value of y . Indeed, we may solve symbolically for x , as in the following example.

Example 9: Let $y = f(x) = -2x - 5$. Solving for x in terms of y ,

$$x = -\frac{1}{2}(y + 5) = -\frac{1}{2}y - \frac{5}{2}.$$

This equation defines x as a linear function of y with rate of change $-\frac{1}{2}$. This linear function is called the inverse of f and is denoted by f^{-1} . The two equations $y = f(x)$ and $x = f^{-1}(y)$ are equivalent, i.e., they have the same meaning and one is satisfied whenever the other is.

The steps in Example 7 can be carried out in general. If $f(x) = mx + b$ is a linear function, we find the inverse function by setting $y = f(x)$ and solving symbolically for $x = f^{-1}(y)$. The result is

$$f^{-1}(y) = \frac{1}{m}(y - b) = \frac{1}{m}y - \frac{b}{m}.$$

Notice that the rate of change of f^{-1} is the reciprocal of the rate of change of f .

Exercise 13: Find the inverses of the linear functions in Examples 1-4.

Example 10: You invest \$20,000 in bonds that draw simple interest at a rate of 5% per year. You would like to have a schedule that tells you when your accumulated fortune will reach various levels. How would you construct one?

Solution: The value of your investment after t years is given in Example 3 above. Notice that t does not have to be a whole number of years. The value of P is 20,000 and the value of r is .05. Therefore, the value of the investment after t years is

$$V = 2,0000(1 + .05t) = 20,000 + 1,000t.$$

This expresses V as a function of t . However, to construct a wealth schedule we want to express t as a function of V , i.e. we want to find the inverse function. But this is easy. Simply solve the equation above for t and put the resulting formula into your calculator or spreadsheet program. The formula is:

$$t = \frac{V - 20,000}{1000}.$$

This is the inverse function.

There is a simple relationship between the graph of f and the graph of f^{-1} . It is illustrated by the following figures. The picture on the left is the graph of the linear function $y = f(x)$. The one on the right is the graph of the inverse function $x = f^{-1}(y)$ with the values of x plotted on the vertical axis and the values of y on the horizontal axis. In other words, we obtain the graph of f^{-1} from the graph of f by flipping it about the diagonal line $y = x$, indicated here by the dashes.

