SEQUENCES:

A function f whose domain is the set of positive integers is called a *sequence.* The values

$$f(1),\ f(2),\ f(3), \dots,\ f(n),\ \dots$$

are called the *terms* of the sequence; $f(1)$ is the first term, $f(2)$ is the second term, $f(3)$ is the third term, . . . , $f(n)$ is the nth term, and so on.

In treating sequences, it is customary to use subscript notation instead of functional notation. Thus, for a given sequence f, $f(1)$ is denoted by a_1, $f(2)$ by a_2, $f(3)$ by a_3, $\dots$, and $f(n)$ by a_n. In this notation, we represent the sequence as

$$a_1, a_2, a_3, \dots, a_n, \dots .$$

Examples 2.2: Write the first four terms, the ninth term and the twentieth term of the sequence whose nth term is:

(1) $a_n = \dfrac{n+1}{n}$

(2) $a_n = \dfrac{n}{n^2+1}$

(3) $a_n = (-1)^n$

(4) $a_n = (-1)^n n^2$

Solutions:

To find the first four terms we substitute, successively, $n = 1, 2, 3, 4$ in the formula for a_n. The ninth and twentieth terms are found by substituting $n = 9$ and $n = 20$ in the formula for a_n. The results are:

	First four terms	Ninth term	Twentieth term
(1)	$2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4}$	$\frac{10}{9}$	$\frac{21}{20}$
(2)	$\frac{1}{2},\ \frac{2}{5},\ \frac{3}{10},\ \frac{4}{17}$	$\frac{9}{82}$	$\frac{20}{401}$
(3)	$-1.\ 1, -1,\ 1$	-1	1
(4)	$-1,\ 4,\ -9,\ 16$	-81	400 ■

Rather than giving an explicit formula for a_n, sequences are sometimes specified by stating a method for finding the terms. One such method is called a *recursion formula* or *recurrence relation*. A recursion formula is a formula that gives a_{n+1} in terms of one or more of the terms that precede a_{n+1}. The starting value, or values, must also be given. We illustrate with some examples.

Examples 2.3: Find the first four terms and the nth term for the sequence specified by:

(1) $a_1 = 3$ and $a_{n+1} = 2a_n$, $k = 1, 2, 3, \ldots$.

(2) $a_1 = 1$ and $a_{n+1} = (n+1)a_n$, $k = 1, 2, 3, \ldots$.

Solutions:

(1) We are given $a_1 = 3$. From the recursion formula

$$\begin{aligned} a_2 &= 2a_1 = 2\cdot 3, \\ a_3 &= 2a_2 = 2\cdot 2\cdot 3 = 2^2\cdot 3, \\ a_4 &= 2a_3 = 2\cdot 2^2\cdot 3 = 2^3\cdot 3. \end{aligned}$$

Based on this pattern, we conclude that $a_5 = 2^4\cdot 3$, $a_6 = 2^5\cdot 3$, and, in general $a_n = 2^{n-1}\cdot 3$.

(2) Proceeding as in (1),

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 2a_1 = 2\cdot 1, \\ a_3 &= 3a_2 = 3\cdot 2\cdot 1, \\ a_4 &= 4a_3 = 4\cdot 3\cdot 2\cdot 1 \end{aligned}$$

Based on this pattern, we conclude that $a_5 = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$, $a_6 = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$, and, in general, $a_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$. * ■

Example 2.4: The sequence defined recursively by:

$$a_1 = a_2 = 1 \quad \text{and} \quad a_{n+2} = a_n + a_{n+1}$$

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* The product of the first n natural numbers is called n *factorial* and is denoted by $n!$; $1!=1$, $2!=2\cdot 1$, $3!=3\cdot 2\cdot 1$, $4!=4\cdot 3\cdot 2\cdot 1$, etc.

is known as the *Fibonacci sequence*. The terms of the sequence are called the *Fibonacci numbers*. The first nine Fibonacci numbers are

1, 1, 2, 3, 5, 8, 13, 21, 34.

It can be shown that

$$a_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}.$$

The Fibonacci numbers appear in a wide variety of applications ranging from the biological sciences to art and architecture. The sequence of ratios of consecutive Fibonacci numbers approaches the number

$$\frac{\sqrt{5}-1}{2}$$

which is the so-called *golden ratio* used by the Greeks thousands of years ago.

Here is another example of a sequence defined by a method for finding the nth term.

Example 2.5: List the first six terms of the sequence whose nth term a_n is the nth prime number.

Solution: $a_1 = 2,\ a_2 = 3,\ a_3 = 5,\ a_4 = 7,\ a_5 = 11,\ a_6 = 13$. It is interesting to note that there is no known formula for a_n. However, this is a perfectly well defined sequence since a_n is defined for each positive integer *n*. ■

In treatments of sequences it is common to make statements such as "consider the sequence whose first four terms are

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots\ ."$$

It is left to the reader: (1) to discover the pattern established by the terms, and (2) to determine a formula for the general term a_n. In this case, the pattern appears to suggest that $a_5 = 1/5,\ a_6 = 1/6$, and, in general, $a_n = 1/n$. However, consider the sequence defined by

$$a_n = \frac{1}{n} + (n-1)(n-2)(n-3)(n-4)\left[\frac{29}{24} - \frac{1}{24 \cdot 5}\right].$$

As you can verify, $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/3$, $a_4 = 1/4$ and $a_5 = 29$. The point of this example is: listing the first few terms of a "sequence" does not, in fact, determine a specific sequence. To determine a specific sequence you must be given a formula for the nth term or you must be given some method for determining the nth term (e.g., a recursion formula). Statements such as "determine the sequence whose first four terms are

$$a_1 \;\; a_2, \;\; a_3, \;\; a_4, \;\; \ldots \text{"}$$

must be qualified by a statement such as "assume that pattern continues as indicated."

Examples 2.6:

(1) The first four terms of a sequence a_n are given. (a) assume that the pattern continues as indicated and find the general formula for a_n; and (b) find a formula for a_n in which $a_5 = \pi / 2$.

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \;\; \ldots$$

(2) The first five terms of a sequence a_n are given. (a) assume that the pattern continues as indicated and find the general formula for a_n; and (b) find a formula for a_n in which $a_6 = 76$.

$$2, -\frac{5}{2}, \frac{10}{3}, -\frac{17}{4}, \frac{26}{5}, \ldots$$

Solutions:

(1) (a) $a_n = \dfrac{2n-1}{2n}$

(b) $a_n = \dfrac{2n-1}{2n} + (n-1)(n-2)(n-3)(n-4)\left[\dfrac{\pi}{48} - \dfrac{11}{12(24)}\right]$

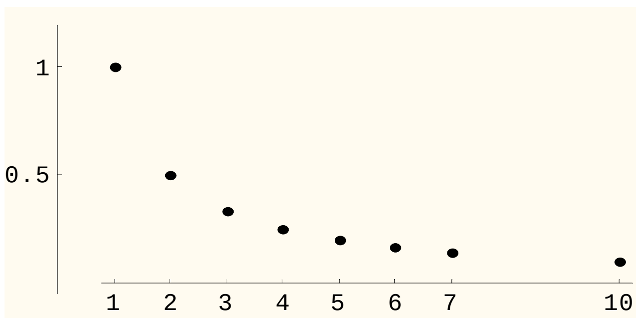
(2) (a) $a_n = (-1)^{n+1} \dfrac{n^2+1}{n}$

(b) $a_n = (-1)^{n+1} \frac{n^2+1}{n} + (n-1)(n-2)(n-3)(n-4)(n-5)\left[\frac{76}{120} + \frac{37}{720}\right]$ ■

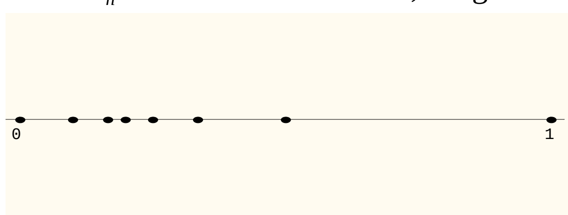
Limits of sequences: Let a_n be a given sequence. We are often interested in the behavior of the terms as n gets larger and larger, symbolized by $n \to \infty$. For example, consider the sequence $a_n = 1/n$. Since

$$\frac{1}{n+1} < \frac{1}{n}, \qquad a_{n+1} < a_n$$

for all n; the terms are decreasing in value. Also, if n is large (say 1,000,000, or 10,000,000), $1/n$ is close to 0. Thus, we conclude that the terms of the sequence $a_n = 1/n$ are decreasing and tending to 0. Here is a graph of the sequence



If we simply plot the values $a_n = 1/n$ on the real line, we get



and we can see that the terms are "headed" toward 0; they never "pass" zero because $1/n > 0$ for every positive integer n. Based on this behavior, we say that "the limit of $1/n$ as n tends to infinity is 0. This is symbolized by

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{or by} \qquad \frac{1}{n} \to 0 \quad as \quad n \to \infty.$$

In general, if the terms of the sequence a_n approach a number L as $n \to \infty$, then L is called the *limit* of the sequence a_n. As suggested in the example above, this is symbolized by

$$\lim_{n\to\infty} a_n = L \quad \text{or by} \quad a_n \to L \quad as \quad n\to\infty \ .$$

Examples 2.7: Determine whether the given sequence has a limit as $n \to \infty$.

(1) $a_n = \dfrac{n+1}{n}$

(2) $a_n = \dfrac{n-1}{n^2}$

(3) $a_n = (-1)^n$

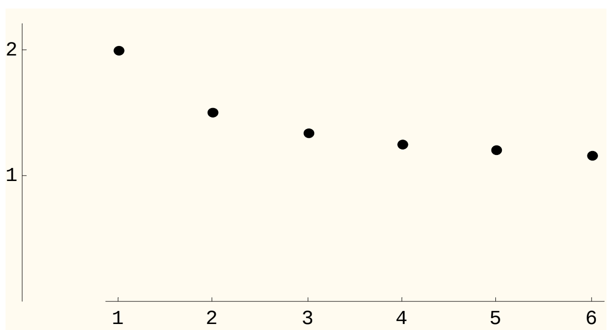
(4) $a_n = (-1)^n n^2$

Solutions:

(1) We write out the first several terms of the sequence to get an idea of its behavior:

$$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots \ .$$

It appears that the terms are getting closer and closer to 1.



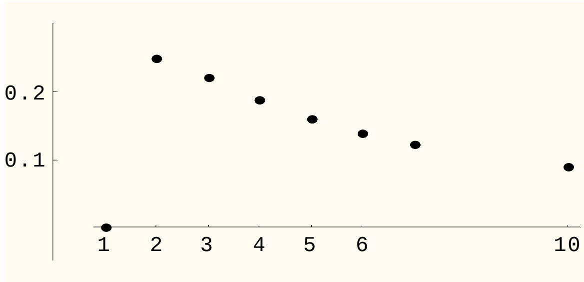
We can justify this conclusion by noting that

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n}.$$

For large n, a_n is close to 1 since $1/n$ is close to 0; $a_n \to 1 \ as \ n \to \infty$.

(2) As in (1), we write out the first several terms

$$0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \dots \ .$$



It appears that the denominators of these fractions are growing much faster than the numerators and so we guess that $a_n \to 0$ *as* $n \to \infty$. This conclusion is justified by writing

$$a_n = \frac{n-1}{n^2} = \frac{n}{n^2} - \frac{1}{n^2} = \frac{1}{n} - \frac{1}{n^2}$$

and noting that $1/n \to 0$ *and* $1/n^2 \to 0$ *as* $n \to \infty$. Therefore $a_n \to 0$ *as* $n \to \infty$.

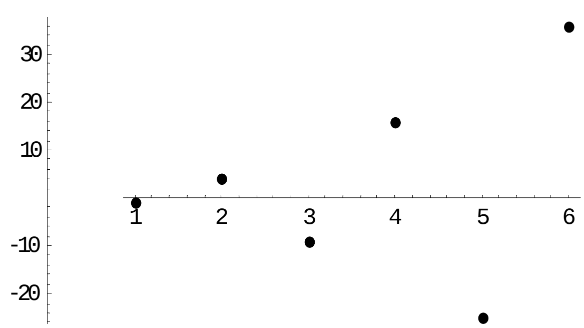
(3) Writing out the first several terms, we have

$$-1, 1, -1, 1, -1, \ldots .$$

The terms simply "oscillate" between -1 and 1; the terms a_n are not getting close to <u>one</u> number L so the limit does not exist.

(4) The first several terms of this sequence are

$$-1, 4, -9, 16, -25, 36, \ldots .$$



The terms get arbitrarily large in absolute value and "oscillate" between positive and negative; the sequence does not have a limit. ■

Two special sequences: We'll conclude this section on sequences by considering two special types of sequences.

Arithmetic sequences: An *arithmetic sequence* is a sequence in which the difference of successive terms is a constant d. That is, a sequence a_n is an arithmetic sequence if

$$a_{n+1} - a_n = d$$

for every positive integer n. The number d is called the *common difference* for the sequence. Note that an arithmetic sequence is defined by a recursion formula. Arithmetic sequences are also called *arithmetic progressions.*

Examples 2.8: Determine whether the given sequence is arithmetic. If it is, give the common difference.

(1) 2, 5, 8, 11, . . . , $3n-1, \ldots$.

(2) 1, 4, 9, 16, . . ., n^2, ...

(3) 22, 18, 14, . . . , $-4n+26, \ldots$

Solutions:

(1) $a_{n+1} - a_n = 3(n+1) - 1 - [3n-1] = 3n + 3 - 1 - 3n + 1 = 3$. The sequence is arithmetic; $d = 3$.

(2) $a_2 - a_1 = 4 - 1 = 3$; $a_3 - a_2 = 9 - 4 = 5$. The sequence is not arithmetic.

(3) $a_{n+1} - a_n = -4(n+1) + 26 - [-4n + 26] = -4n - 4 + 26 + 4n - 26 = -4$. The sequence is arithmetic with $d = -4$. ■

Suppose a_n is an arithmetic sequence with first term $a_1 = a$ and common difference d. We will use the recursion formula to find a formula for a_n. From the definition, we know that $a_{n+1} = a_n + d$. Therefore

$$a_2 = a + d,\ a_3 = a_2 + d = a + 2d,\ a_4 = a_3 + d = a + 3d,\ a_5 = a_4 + d = a + 4d, \ldots$$

and we conclude that

$$a_n = a + (n-1)d \ .$$

Example 2.9: Find the twelfth term of the arithmetic sequence whose first three terms are
1, 5, 9, ...

Solution: $a_1 = a = 1$ and $d = 9 - 5 = 5 - 1 = 4$. Therefore,

$$a_{12} = a + (n-1)d = 1 + 11(4) = 45 \ . \qquad \blacksquare$$

Geometric sequences: A *geometric sequence*, or *geometric progression*, is a sequence in which the ratio of successive terms is a nonzero constant r. That is, a_n is a geometric sequence if and only if

$$\frac{a_{n+1}}{a_n} = r$$

for every positive integer n. The number r is called the *common ratio.* Note that a geometric sequence is defined by a recursion formula:

$$a_{n+1} = r \cdot a_n \ .$$

Examples 2.10:

(1) The sequence 8, 4, 2, 1, . . . , is a geometric sequence. Find the common ratio and give the fifth term of the sequence.

(2) The sequence $5, \ -\frac{5}{2}, \ \frac{5}{4}, \ -\frac{5}{8}, \ldots$ is a geometric sequence. Find the common ratio and give the sixth term.

Solutions:

(1) The common ratio is $r = \frac{a_2}{a_1} = \frac{4}{8} = \frac{1}{2}$. The fifth term is $a_5 = \frac{1}{2}a_4 = \frac{1}{2} \cdot 1 = \frac{1}{2}$.

(2) The common ratio is $r = \frac{a_2}{a_1} = \frac{-5/2}{5} = -\frac{1}{2}$. The sixth term is the fifth term times $-1/2$; the fifth term is $(-1/2)(-5/8) = 5/16$. Therefore, the sixth term is $-5/32$.

Suppose that a_n is a geometric sequence with first term $a_1 = a$. We'll use the recursion formula to determine an expression for the general term a_n:

$$a_2 = a \cdot r,\ \ a_3 = a_2 \cdot r = a \cdot r^2,\ \ a_4 = a_3 \cdot r = a \cdot r^3,\ \ a_5 = a_4 \cdot r = a \cdot r^4, \ldots\ .$$

Based on this pattern, we conclude that

$$a_n = a \cdot r^{n-1}.$$

Example 2.11: Find the seventh term of the geometric sequence whose first three terms are $2,\ -\frac{1}{2},\ \frac{1}{8},\ \ldots$.

Solution: The first term $a = 2$ and the common ratio r is

$$\frac{-1/2}{2} = -\frac{1}{4}.$$

Therefore, by the formula derived above, the seventh term is:
$a_7 = 2(-1/4)^6 = \frac{2}{4^6} = \frac{1}{2048}$.

SERIES

Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a given sequence. Suppose we are asked to add up all the terms of the sequence. That is, suppose we are asked to calculate

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots\ .$$

Since we can only add a finite number of numbers, it would appear that adding up *all* the terms of a sequence is an impossible task. However, there are instances where we can assign a value, a number, to an "infinite sum." We'll explore that possibility here.

A sum of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots\ ,$$

where a_n is a given sequence, is called a *series* (also called an *infinite series*). Since we can add up any finite collection of numbers, we'll form a new sequence S_n by adding up terms of the given sequence in a systematic manner.

Let

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$
$$\vdots$$

This new sequence is called the *sequence of partial sums* of the given sequence a_n.

The questions we want to address are: (1) Can we find a formula for the sequence S_n of partial sums derived from some given sequence a_n? (2) If the answer to (1) is "yes", does S_n have a limit, say S, as $n \to \infty$?

If the answers to (1) and (2) are "yes", then we say that the sum of all the terms of the sequence a_n is S; we've added up infinitely many numbers and gotten a number! Here are some examples.

Examples 2.12:

1. Let a_n be the sequence defined by $a_n = (-1)^{n+1}$. This is the sequence

$$1, -1, 1, -1, \ldots$$

The sequence of partial sums is:

$$S_1 = 1$$
$$S_2 = 1 - 1 = 0$$
$$S_3 = 1 - 1 + 1 = 1$$
$$S_4 = 1 - 1 + 1 - 1 = 0$$
$$\vdots$$

Based on these results, we conclude that $S_n = 1$ when n is odd and $S_n = 0$ when n is even. The sequence S_n can be represented by

$$S_n = \begin{Bmatrix} 1 \text{ when } n \text{ is odd} \\ 0 \text{ when } n \text{ is even} \end{Bmatrix} \quad \text{or by} \quad S_n = \frac{1 + (-1)^{n+1}}{2}.$$

Does S_n have a limit as $n \to \infty$?

2. Let a_n be the sequence defined by $a_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$.

The first three terms of the sequence of partial sums is:

$$S_1 = 1 - \frac{1}{2}$$
$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$
$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$
$$\vdots$$

What is S_n for every positive integer n? Does S_n have a limit as $n \to \infty$?

3. Let a_n be the sequence defined by $a_n = n$. This is the sequence of positive integers

$$1,\ 2,\ 3,\ 4,\ \ldots,\ n,\ \ldots\ .$$

The sequence of partial sums is:

$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = 1 + 2 + 3 = 6$$
$$S_4 = 1 + 2 + 3 + 4 = 10$$
$$\vdots$$

What is S_n for every positive integer n? Does S_n have a limit as $n \to \infty$?

Solutions:

1. S_n does not have a limit; the sequence "oscillates" between 0 and 1.

2. $S_n = 1 - \dfrac{1}{n}$; $S_n \to 1$ as $n \to \infty$. The "infinite" series has a finite sum:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots = 1.$$

3. The sequence 1, 2, 3, 4, …, is an arithmetic sequence with $a_1 = 1$ and $d = 1$.

$$S_n = 1 + 2 + \ldots + (n-1) + n$$
$$S_n = n + (n-1) + \ldots + 2 + 1$$

Adding these two equations, we get

$$2S_n = (n+1) + (n+1) + \ldots + (n+1) \qquad [n \text{ terms}]$$
$$= n(n+1).$$

Solving for S_n, we get the result

$$S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

This says that the sum of the first n positive integers is $\dfrac{n(n+1)}{2}$.

The sequence of partial sums does not have a limit; $S_n \to \infty$ as $n \to \infty$.

Arithmetic and geometric series: Let $a_1, a_2, a_3, a_4, \ldots$ be an arithmetic sequence with first term $a_1 = a$ and common difference d. Then,

$$a_1 = a,\ a_2 = a + d,\ a_3 = a + 2d,\ a_4 = a + 3d, \ldots$$

and

$$a + [a+d] + [a+2d] + \ldots + [a + (n-1)d] + \ldots$$

is an arithmetic series. The sequence of partial sums is:

$$S_1 = a$$
$$S_2 = a + (a+d) = 2a + d$$
$$S_3 = a + (a+d) + (a+2d) = 3a + (1+2)d$$
$$S_4 = a + (a+d) + (a+2d) + (a+3d) = 4a + (1+2+3)d$$
$$\vdots$$
$$S_n = na + [1 + 2 + 3 + \ldots + (n-1)]d$$
$$\vdots$$

Using our result in (3) above, $1 + 2 + 3 + \ldots + (n-1) = \dfrac{(n-1)n}{2}$ (replace n by $n-1$ in the formula) and

$$\boxed{S_n = na + \frac{n(n-1)}{2}d \quad \text{or} \quad S_n = \frac{n}{2}[2a + (n-1)d]}.$$

This is a formula for the sequence of partial sums of an arithmetic series.

Since $a_n = a + (n-1)d$, it follows that $2a + (n-1)d = a + [a + (n-1)d] = a + a_n$. Therefore, S_n can also be written as

$$\boxed{S_n = \frac{n}{2}(a + a_n).}$$

Now suppose that $a_1, a_2, a_3, a_4, \ldots$ is a geometric sequence with first term $a_1 = a$ and common ratio r. Then

$$a_1 = a,\ a_2 = ar,\ a_3 = ar^2,\ a_4 = ar^3,\ \ldots$$

and

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$$

is a geometric series. The sequence of partial sums is:

$$\begin{aligned}
S_1 &= a \\
S_2 &= a + ar \\
S_3 &= a + ar + ar^2 \\
S_4 &= a + ar + ar^2 + ar^3 \\
&\vdots \\
S_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\
&\vdots
\end{aligned}$$

Note first that if $r = 1$, then

$$S_n = \underset{(n\ a\text{'s})}{a + a + a + \cdots + a} = na.$$

Now assume that $r \neq 1$. Then

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

and

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$$

Subtracting the second equation from the first gives

$$S_n - rS_n = a - ar^n \quad \text{and} \quad (1-r)S_n = a - ar^n .$$

Therefore,

$$\boxed{S_n = \frac{a - ar^n}{1-r} \quad \text{provided} \;\; r \neq 1.}$$

This is a formula for the sequence of partial sums of a geometric series.

Let's look at the behavior of S_n as $n \to \infty$. We can rewrite our formula for S_n as

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

It is possible to show that if $1 < r < 1$, then $r^n \to 0$ as $n \to \infty$, and it follows that

$$\frac{ar^{n-1}}{1-r} \to 0 \quad \text{and} \quad S_n \to \frac{a}{1-r}.$$

Therefore, if the common ratio r of a geometric series satisfies $-1 < r < 1$, (i.e., $|r| < 1$), then we define the (finite) sum of the infinite geometric series

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$$

to be

$$\boxed{S = \frac{a}{1-r}, \quad |r| < 1.}$$

Examples 2.13:

1. Find the sum of the first 8 terms of the arithmetic sequence 3, 7, 11, 15, . . .

2. Find the sum of the first 6 terms of the geometric sequence $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$.

   Does the infinite series

$$3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$$

have a finite sum?

Solutions:

1. a = 3 and d = 4. Using the first arithmetic series formula, we have

$$S_8 = \frac{8}{2}[2(3)+(8-1)4] = 4(6+28) = 136.$$

Using the second formula, $a_8 = 3+(8-1)4 = 31$ and

$$S_8 = \frac{8}{2}[3+31] = 4(34) = 136.$$

2. $a = 3$ and $r = \frac{1}{2}$. Therefore

$$S_6 = \frac{3\left[1-\left(\frac{1}{2}\right)^6\right]}{1-\frac{1}{2}} = \frac{3\left[1-\frac{1}{64}\right]}{\frac{1}{2}} = 6\left[1-\frac{1}{64}\right] = 6\left(\frac{63}{64}\right) = \frac{189}{32}.$$

Since $|r| = \frac{1}{2} < 1$, the infinite geometric series has the finite sum

$$S = \frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = \frac{3}{\frac{1}{2}} = 6.$$

Exercises:

Find the first five terms and the eighth term of the following sequences.

1. $a_n = 10-2n$

2. $a_n = \frac{3}{1-2n}$

3. $a_n = \frac{2n-4}{n^2+1}$

4. $a_n = 8 + \frac{1}{n}$

5. $a_n = 6$

6. $a_n = 2 + (-1)^n$

7. $a_n = 2 + (0.1)^n$

8. $a_n = (-1)^{n-1} \frac{n+1}{2n}$

9. $a_n = \frac{1 + (-1)^{n+1}}{2}$

10. $a_n = \frac{2^n}{n^2 + 2}$

Find the first five terms of the sequence defined by the given recursion formula. If possible, find a formula for a_n.

11. $a_1 = 1,\ a_{n+1} = \frac{a_n + 1}{2}$

12. $a_1 = 1,\ a_{n+1} = \frac{1}{n+1} a_n$

13. $a_1 = 2,\ a_{n+1} = 2a_n + 2$

14. $a_1 = 1,\ a_{n+1} = a_n + 2n + 1$

15. $a_1 = 1,\ a_{n+1} = a_n + a_{n-1} + \cdots + a_1$

16. $a_1 = 1,\ a_{n+1} = 2a_n + 1$

Determine whether the given sequence has a limit. If it does, give the limit.

17. $a_n = \frac{n-1}{n}$

18. $a_n = \frac{n^2}{n+1}$

19. $a_n = \frac{n-1}{n^2}$

20. $a_n = \frac{n+(-1)^n}{n}$

21. $a_n = \frac{2^n}{4^n+1}$

22. $a_n = \frac{4n}{\sqrt{n^2+1}}$

Determine whether the indicated sequence can be the first three terms of an arithmetic or geometric sequence, and, if so, find the common difference or common ratio and the general term a_n.

23. $-11, -16, -21, \ldots$

24. $1, 4, 9, \ldots$

25. $2, -4, 8, \ldots$

26. $7, 6.5, 6, \ldots$

27. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$

28. $3, 3, 3, \ldots$

29. $\frac{1}{2}, \frac{1}{4}, \frac{1}{9}, \ldots$

30. $7, 14, 28, \ldots$

Find the tenth term and the nth term of the given arithmetic sequence.

31. 2, 6, 10, 14, . . .

32. 11, 9, 7, . . .

33. 3, 2.7, 2.4, 2.1, . . .

34. 7, 6.5, 6, . . .

Let $a_1, a_2, a_3, \ldots$ be an arithmetic sequence. Find the indicated quantities.

35. $a_1 = a = 7,\ d = 2;$ find a_5 and S_{10}.

36. $a_1 = a = 2,\ d = -3;$ find a_{10} and S_{20}.

37. $a_1 = a = 5,\ d = 0.1;$ find a_{10} and S_{20}.

38. $a_1 = a = 7/3,\ d = -2/3;$ find a_{16} and S_{16}.

Find the tenth term and the nth term of the given geometric sequence.

39. 8, 4, 2, 1, . . .

40. 2, 6, 18, . . .

41. 4, 1.2, 0.36, . . .

42. 4, –6, 9, –13.5, . . .

Let $a_1, a_2, a_3, \ldots$ be a geometric sequence. Find the indicated quantities.

43. $a_1 = a = 3,\ r = -2;$ find a_6 and S_6.

44. $a_1 = a = 2,\ r = 1/3;$ find a_4 and S_4.

45. $a_1 = a = 1,\ r = 3/2;$ find a_4 and S_4.

46. $a_1 = a = 50,\ r = 0.5;$ find a_{10} and S_{10}.

Determine whether the geometric series has a finite sum. If it does, find it.

47. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

48. 50 + 25 + 12.5 + . . .

49. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$

50. $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\ldots$