Functions defined on intervals: In this section we consider functions $f : X \to Y$ where X is an interval of real numbers or a union of intervals, and Y is the set of real numbers. Our purpose here is to introduce the basic ideas and terminology that will be used in the next two sections to study specific classes of functions. Our discussion will emphasize the connection between functions defined on intervals and their graphs.
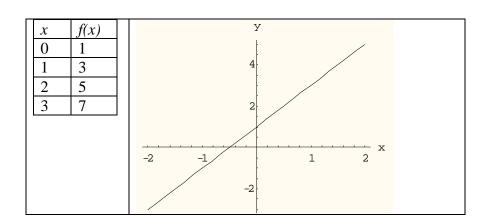
Graphs: Let f be a function with domain X, an interval or union of intervals. Recall that the *graph of f* is the set of all points $(x, f(x))$ in the coordinate plane. The graph of f can also be described as the set of all points (x, y) such that $y = f(x)$, $x \in X$. Thus, the graph of f is the same as the graph of the equation $y = f(x)$. For the types of functions that we will be considering, the graph of f will be a curve in the plane. Until we learn more sophisticated methods, we will draw graphs simply by plotting points. We'll have to make sure that we plot enough points to see what the entire graph looks like.
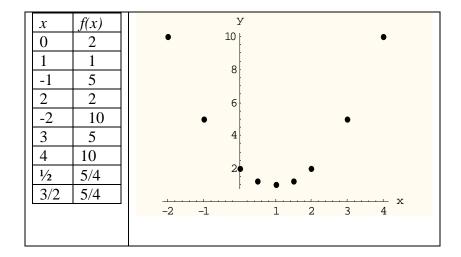
Examples 3.1: Draw the graph of :

a. $f(x) = 2x + 1$.

b. $f(x) = x^2 - 2x + 2$.

Solutions:

a.

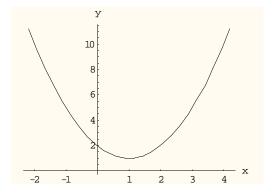| x	f(x)
0	1
1	3
2	5
3	7

The points appear to lie on a straight line. Indeed they do. This will be proved in the next section. The function $f(x) = 2x + 1$ is an example of a *linear function.* Of course, if we know in advance that f is a function whose graph is a straight line, then we only need to plot two points; the graph will be the line determined by the two points.

b. To graph $f(x) = x^2 - 2x + 2$ we'll first plot some points:

x	$f(x)$
0	2
1	1
-1	5
2	2
-2	10
3	5
4	10
½	5/4
3/2	5/4

Following the pattern suggested by these points, we get the graph

The function $f(x) = x^2 - 2x + 2$ is an example of a *quadratic function.* The graph of a quadratic function is a *parabola.* Quadratic functions and efficient methods for drawing their graphs will be taken up in detail later in this module.
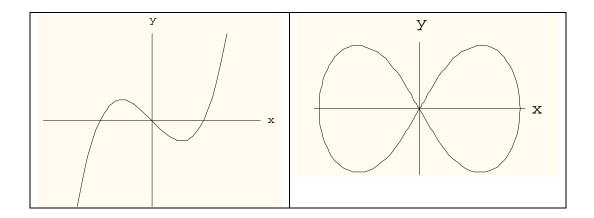
The vertical line test: For the types of functions that we are considering here, graphs are curves in the plane. This raises a converse question: Suppose we are given a curve C in the plane. Is it the graph of some function f?

Since a curve in the plane is a set S of ordered pairs (x,y) – the coordinates of the points that lie on the curve – we can apply the "ordered pairs" definition of "function" given above: A set S of ordered pairs (x, y) is a function if and only if no two distinct ordered pairs in S have the same first coordinate and *different* second coordinates. That is, (a, y_1) and (a, y_2) are in S if an only if $y_1 = y_2$. It now follows that if C is a curve in the plane and if there are two distinct ordered pairs (a, y_1) and (a, y_2) that lie on C, then C cannot be the graph of the function.

If $y_1 \neq y_2$, then the points (a, y_1) and (a, y_2) are distinct points on the vertical line through the point $(a,0)$. Thus, a curve C is not the graph of a function if some vertical line intersects C in more than one point. Equivalently, a curve C in the plane *is* the graph of a function if *no* vertical line intersects C more than once.

Examples 3.2:

Pictured below are two curves. The curve on the left is the graph of a function; the curve on the right is not the graph of a function because there are vertical lines that intersect the curve more than once.
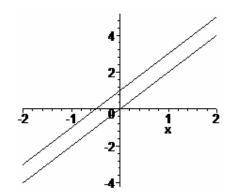

Translations and scaling. Before discussing the specific types of functions that we will be treating in this module, we give some general information about functions and their graphs.

Vertical translations: Let f be a given function and let b be a positive number. What is the relationship between the graphs of $y = f(x)$, $y = f(x) + b$ and $y = f(x) - b$? Consider the following examples.
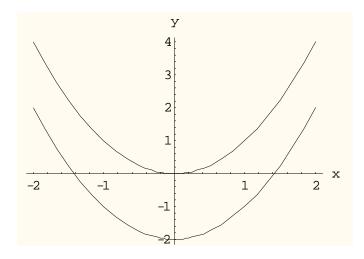
Examples 3.3:

1. $y = 2x, \quad y = 2x + 1$

The graph of $y = 2x + 1$ is the graph of $y = 2x$ translated up 1 unit.
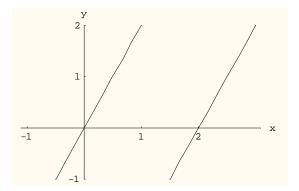
2. $y = x^2, \quad y = x^2 - 2$

The graph of $y = x^2 - 2$ is the graph of $y = x^2$ translated down 2 units.

In general, the graph of $y = f(x) + b$ is the graph of $y = f(x)$ translated up b units and the graph of $y = f(x) - b$ is the graph of $y = f(x)$ translated down b units.

Horizontal translations: Here we consider the relationship between the graphs of $y = f(x)$, $y = f(x - c)$ and $y = f(x + c)$ where c is a positive number. Again, we consider some specific examples to illustrate the general result.
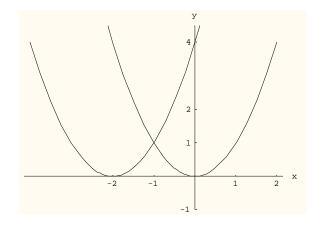
Examples 3.4:

1. $y = 2x, \quad y = 2(x - 2)$

The graph of $y = 2(x - 2)$ is the graph of $y = 2x$ translated to the right 2 units.
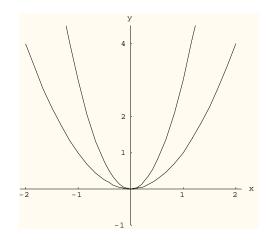
2. $y = x^2, \quad y = (x + 2)^2.$

The graph of $y = (x + 2)^2$ is the graph of $y = x^2$ translated to the left 2 units.
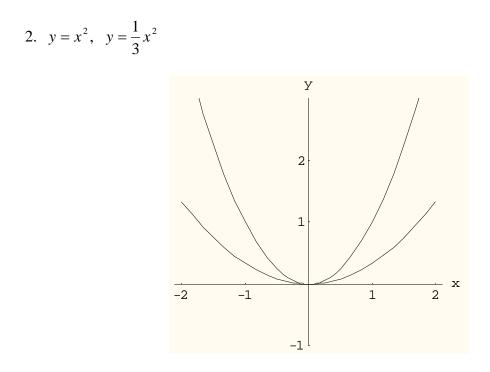
In general, the graph of $y = f(x - c)$ is the graph of $y = f(x)$ translated c units to the right and $y = f(x + c)$ is the graph of $y = f(x)$ translated c units to the left.

Scaling: Let f be a given function and let c be a positive number. Here we look at the effect of multiplying f by c. That is, we want to compare the graphs of $y = f(x)$ and $y = c \cdot f(x)$, (assume that $c \neq 1$). As above, we illustrate the general result by means of some examples.

Examples 3.5:

1. $y = x^2 \quad y = 3x^2.$
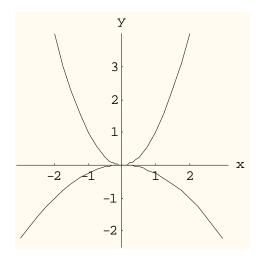
2. $y = x^2$, $y = \dfrac{1}{3}x^2$

In general, if $c > 1$, then $y = c \cdot f(x)$ stretches the graph of $y = f(x)$ vertically by c units. If $0 < c < 1$, then $y = c \cdot f(x)$ compresses the graph of $y = f(x)$ by c units.

Finally, suppose we multiply f by a negative number c? In addition to stretching or compressing the graph of $y = f(x)$ (depending on whether $|c|$ is greater than 1 or less than 1), the graph will be reflected in the x-axis.

Example 3.6:

$y = x^2$, $y = -\dfrac{1}{2}x^2$.

The Elementary Functions.

The functions that we will be studying in this module will be "built up" as sums, differences, products, and quotients of "simple" functions. These functions are part of the class of functions known as the "elementary functions." We will define what we mean by the "elementary functions" at the end of this section.
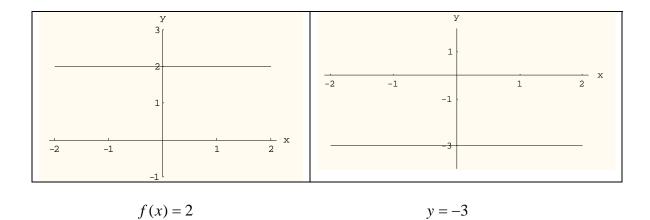
To begin, our "simple" functions will have all real numbers as their domain; that is, the interval $(-\infty, \infty)$. Therefore, until further notice, $X = Y =$ the set of real numbers.

Polynomial functions: The simplest functions are the constant functions: f is a constant function if $f(x) = b$ (constant) for all x. The graph of the constant function $f(x) = b$ (or $y = b$) is the set of all points (x, b) in the plane. This is a horizontal line b units above or below the x-axis depending on the sign of b.
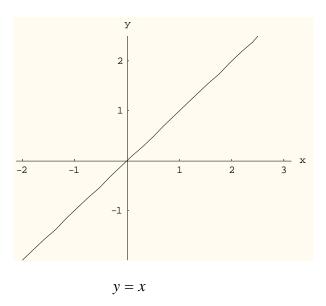
Examples 3.2: Draw the graph of :

(1) $f(x) = 2$.   (2) $y = -3$.

Solutions:

$$f(x) = 2$$

$$y = -3$$
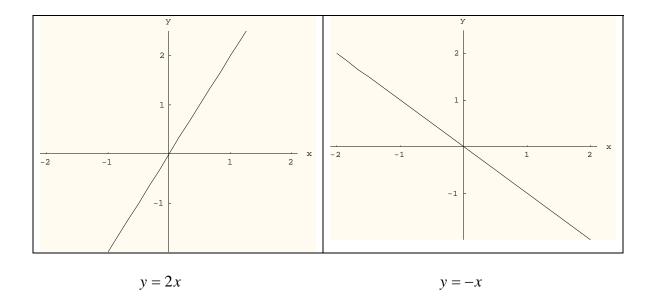
The function $f(x) = x$ is called the *identity function* – the value of f at x is x. The set of all points of the form (x, x) in the plane is the line through the origin that bisects the first and third quadrants.
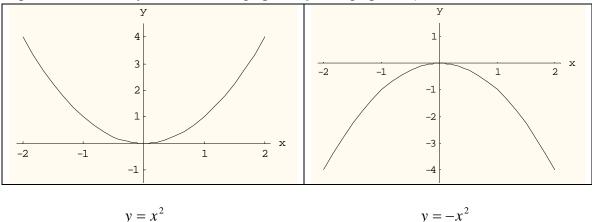
$$y = x$$

The graphs of constant multiples of the identity function, $f(x) = mx$, are lines through the origin. For example, the graphs of $f(x) = 2x$ and $f(x) = -x$ are:

$$y = 2x \qquad\qquad\qquad\qquad y = -x$$

Linear functions, the topic of the next section, are simply sums of constant functions and multiples of the identity function. That is, a linear function is a function of the form $f(x) = mx + b$. The graph of a linear function is a straight line. See the graph of $f(x) = 2x + 1$ given above.

Next we'll consider powers of the identity function. The function $g(x) = x^2$ is the square of the identity function. The graph of g (the graph of $y = x^2$) is

$$y = x^2 \qquad\qquad\qquad\qquad y = -x^2$$

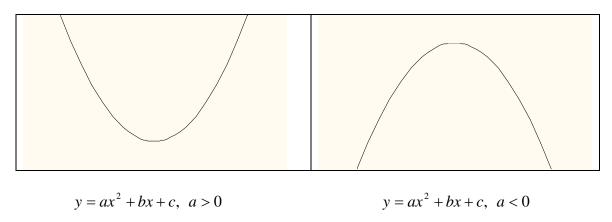This type of curve is called a *parabola*. The function $h(x) = -x^2$ is the reflection of $y = x^2$ in the x-axis. The graphs of constant multiples of $g(x) = x^2$, say $f(x) = ax^2$ have the same general shape as the graph of g or h, depending on the sign of a, and are either steeper or flatter depending on the magnitude of a. See the examples given above.
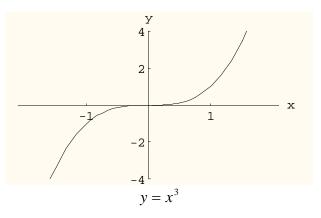
Sums of constant functions, multiples of the identity function and multiples of $g(x) = x^2$ have the form

$$f(x) = ax^2 + bx + c, \quad a \neq 0.$$

These functions are called *quadratic functions.* Quadratic functions and their applications will be studied in detail later in this module. The graph of a quadratic function is a parabola which opens upward if $a > 0$ and opens downward if $a < 0$.

$$y = ax^2 + bx + c, \quad a > 0 \qquad\qquad\qquad y = ax^2 + bx + c, \quad a < 0$$
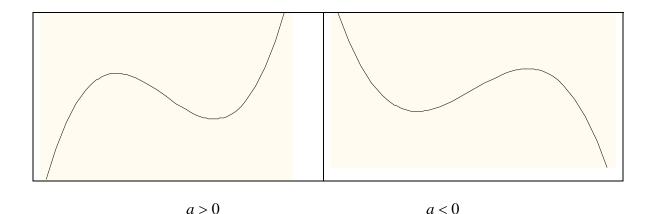
The graph of the function $g(x) = x^3$ (the "cube" of the identity function) is

$$y = x^3$$

Adding constants, multiples of the identity, and multiples of x^2 to multiples of this function, we get functions of the form

$$f(x) = ax^3 + bx^2 + cx + d.$$

These functions are called *cubics.* In general, the graph of a cubic looks like

$$a > 0 \qquad\qquad\qquad a < 0$$

If we continue this process we are led to a general (and important) class of functions called *polynomial functions.* A polynomial function is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \quad a_n \neq 0,$$

where n is a nonnegative integer, called the *degree of p,* and $a_0, a_1, \ldots, a_{n-1}, a_n$ are real numbers called the *coefficients of p.* (Note: We had to switch to subscript notation because if n is large, say $> 22,$ then there are not enough distinct letters to fill all the slots.)

Polynomial functions of degree 0 have the form

$$p(x) = a, \ \ a \neq 0,$$

and are the nonzero constant functions.

Polynomial functions of degree 1 have the form

$$p(x) = ax + b, \ \ a \neq 0,$$

and are called *linear polynomials* or linear functions, as we saw above.

Polynomial functions of degree 2 have the form

$$p(x) = ax^2 + bx + c, \ \ a \neq 0,$$

and are called quadratic polynomials or quadratic functions, a principal topic in the last section of this module.

Rational functions: Now that we have defined polynomial functions, we can go on to the next class of functions, the rational functions. A *rational function* is the quotient of two polynomial functions. Thus, r is a rational function if

$$r(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. The functions

$$F(x) = \frac{1}{x-2}, \qquad G(x) = \frac{2x+1}{x^2+4}, \qquad \text{and} \qquad H(x) = \frac{x}{x^2-1}$$

are examples of rational functions. The relationship between polynomials and rational functions is analogous to the relationship between the integers and rational numbers. For example, just as every integer k can be regarded as a rational number, $k = k/1$, every polynomial p can be regarded as a rational function $p(x) = p(x)/1$.

Up to this point our functions have been defined for all real numbers so we have been assuming that the domain $X = (-\infty, \infty)$. The introduction of rational functions means that we have to be careful about domains. Since division by zero is not defined, the domain of a rational function will not, in general, be all real numbers.

Convention on domains: Given a function f. If the domain of f is not given explicitly or specified implicitly by an application, then the domain is assumed to be the set of all real numbers x such that $f(x)$ is a real number.

According to this convention, the domain of the rational function

$$r(x) = \frac{p(x)}{q(x)}$$

is the set of all real numbers except for those numbers c at which $q(c) = 0$.

Examples 3.3: Determine the domain of each of the following rational functions:

(1) $r(x) = \dfrac{1}{x-2}$ (2) $F(x) = \dfrac{2x+1}{x^2+4}$ (3) $G(x) = \dfrac{x}{1-x^2}$
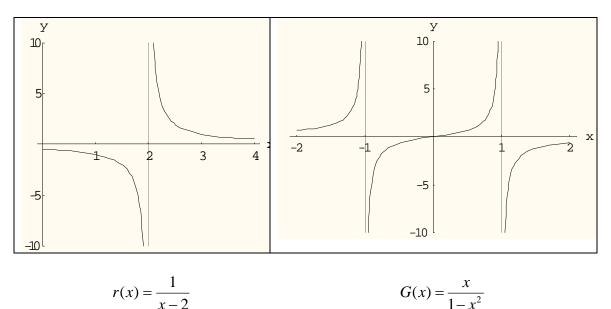
Solutions:

(1) $r(x) = \dfrac{1}{x-2}$: Since $x - 2 = 0$ at $x = 2$ and is nonzero otherwise, the domain of r is the set of all real numbers except $x = 2$, i.e., the domain $X = (-\infty, 2) \cup (2, \infty)$.

(2) $F(x) = \dfrac{2x+1}{x^2+4}$: Since $x^2 + 4 \neq 0$ for all x, the domain of F is all real numbers; $X = (-\infty, \infty)$.

(3) $G(x) = \dfrac{x}{1-x^2}$: Since $1-x^2 = (1-x)(1+x) = 0$ at $x = -1$ and $x = 1$, and is nonzero otherwise, the domain of G is all real numbers except $x = -1$, $x = 1$; $X = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

In order to obtain the graph of a rational function $r(x) = p(x)/q(x)$, we have to examine carefully the behavior of $r(x)$ for values of x near the numbers c where $q(c) = 0$. In particular, if $q(c) = 0$ and $p(c) \neq 0$, then $r(x)$ grows arbitrarily large, either in the positive direction or in the negative direction, as x approaches c. The figure below shows the graphs of $r(x) = 1/(x-2)$ and $G(x) = x/(1-x^2)$. Note the behavior of these functions for values of x near the points at which the respective denominators are 0.

$$r(x) = \frac{1}{x-2} \qquad\qquad G(x) = \frac{x}{1-x^2}$$

Algebraic functions: Sums, differences, products, and quotients of rational functions produce rational functions; the rational functions are said to be *closed* with respect to the algebraic operations of addition, subtraction, multiplication and division.
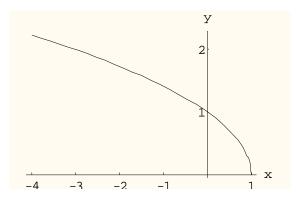
There is yet another algebraic operation, namely taking roots; square roots, cube roots, etc. This operation adds further complexity to finding domains, ranges and graphs. For example, the domain of the function $f(x) = 1 - x$ is all real numbers. Now consider the function

$$g(x) = \sqrt{1-x} \, .$$

Since the square root of a negative number is not a real number, the domain of g must exclude all numbers x such that $1 - x < 0$ that is, all numbers x such that $x > 1$. Thus, by our convention,

$$\text{domain } g = \{x \mid x \leq 1\}.$$

The graph of g is:

In general, the set of *algebraic functions* is the set of functions that can be formed by adding, subtracting, multiplying, dividing and taking roots of rational functions. Some examples of algebraic functions are:

$$f(x) = \sqrt{\frac{x}{4 - x^2}} \quad \text{and} \quad g(x) = \sqrt[3]{x^4 - 2x^3 - 2x + 1}.$$

What about the domain of each of these functions; what about their graphs? These questions do not have simple answers. We'll not attempt graph either of these functions, but it is a good exercise to determine that:

$$\text{domain } f = \{x \mid 0 \leq x < 2 \text{ or } x < -2\} \quad \text{and} \quad \text{domain } g = \text{all real numbers}.$$

Exponential Functions: Let a be a positive number. The *exponential function f with base a* is the function defined by

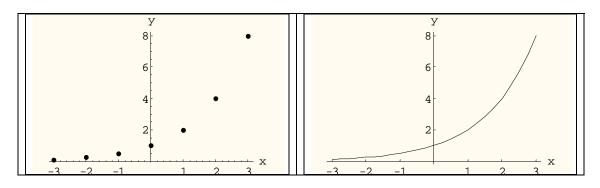$$f(x) = a^x \quad \text{where } x \text{ is any real number}.$$

Examples:

1. Sketch the graph of $f(x) = 2^x$.

The coordinates of some points on the graph are given in the following table.

x	-3	-2	-1	0	1	2	3
2^x	1/8	1/4	1/2	1	2	4	8

We plot these points and then draw graph.

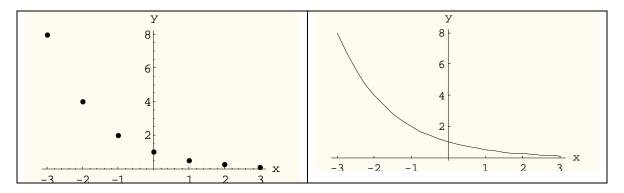2. Sketch the graph of $f(x) = \left(\dfrac{1}{2}\right)^x$

The coordinates of some points on the graph are given in the following table.

x	-3	-2	-1	0	1	2	3
2^x	8	4	2	1	1/2	1/4	1/8

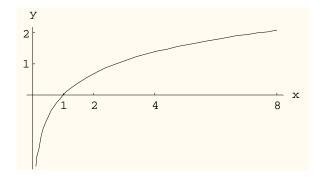We plot these points and then draw the graph.

In general, the graph of $f(x) = a^x$ resembles the graph of $f(x) = 2^x$ if $a > 1$ and it resembles the graph of $f(x) = (1/2)^x$ if $0 < a < 1$. In this context, the graph of $f(x) = 1^x$ is not very interesting; $1^x \equiv 1$ for all x so the graph of f is simply the graph of $y = 1$, a horizontal line 1 unit above the x-axis.

Exponential functions have many important applications to "real-life" problems. For example, money invested at compound interest (as opposed to simple interest) grows exponentially. Radioactive materials decay exponentially.

Logarithm functions: Choose a positive number a, $a \neq 1$, and let f be the exponential function $f(x) = a^x$. The domain of f is the set of all real numbers and the range of f is the set of positive numbers. Moreover, f is a one-to-one function. Therefore it has an inverse function. The inverse function of f is a *logarithm function.* In particular, the inverse of $f(x) = a^x$ is the function g given by

$$g(x) = \log_a x$$

read "the log of x to the base a." The domain of g is the set of positive numbers and the range is the set of all real numbers.

The graph of $g(x) = \log_a x$ is the reflection of the graph of $f(x) = a^x$ in the line $y = x$. For $a > 1$, the graph of $g(x) = \log_a x$ looks like

The relationship between $f(x) = a^x$ and $g(x) = \log_a x$ is as follows:
$$a^x = y \quad \text{if and only if} \quad x = \log_a y.$$

It is this relationship that allows us to solve exponential equations and this is the typical use of logarithm functions in algebra.

Trigonometric functions: The basic trigonometric functions, sine and cosine, are defined in terms of the coordinates of the point on the unit circle that corresponds to a central angle with initial side on the positive x-axis. There are six trigonometric functions in all; the other four – tangent, secant, cotangent and cosecant – are defined in terms of sine and cosine.

We won't go further with the trigonometric functions here because these functions are not included in a treatment of "algebra." We mention them only because the following classes of functions:

1. polynomial functions
2. rational functions
3. algebraic functions
4. exponential and logarithm functions
5. trigonometric functions

constitute what is known as the *elementary functions*.