

Aunt Sarah and the Farm

Purpose:

Participants will explore a situation giving rise to difference of squares, polynomial multiplication, and geometric construction of functions.

Overview:

In small groups, participants will explore a land deal giving rise to multiple opportunities for mathematical modeling. They will have the option of using a spreadsheet to create multiple representations for exploration including graphical, tabular and functional.

TEXES Mathematics Competencies. The beginning teacher:

- II.04.A Uses inductive reasoning to identify, extend, and create patterns using concrete models, figures, numbers, and algebraic expressions.
- II.04.B Formulates implicit and explicit rules to describe and construct sequences verbally, numerically, graphically, and symbolically.
- II.04.D Gives appropriate justification of the manipulation of algebraic expressions.
- II.06.B Demonstrates an understanding of the connections among geometric, graphic, numeric, and symbolic representations of quadratic functions.
- II.06.C Analyzes data and represents and solves problems involving exponential growth and decay.

TEKS Mathematics Objectives. The student is expected to:

- 5.06.A Describe relationships mathematically. The student is expected to select from and use diagrams and number sentences to represent real-life situations.
- 6.04. Generate formulas to represent relationships involving perimeter, area, volume of a rectangular prism, etc., from a table of data.
- 6.05 Formulate an equation from a problem situation.
- 6.8.B Select and use appropriate units, tools, or formulas to measure and to solve problems involving length (including perimeter and circumference), area, time, temperature, capacity, and weight.
- 7.2.C Use models to add, subtract, multiply, and divide integers and connect the actions to algorithms
- 7.4.A Generate formulas involving conversions, perimeter, area, circumference, volume, and scaling.
- 7.4.B Graph data to demonstrate relationships in familiar concepts such as conversions, perimeter, area, circumference, volume, and scaling
- 7.4.C Describe the relationship between the terms in a sequence and their positions in the sequence.

Terms.

Area, perimeter, function, sequence, difference of squares

Materials.

For each small group of participants:

- Transparency
- Activity Sheet for each participant
- Graph paper

For the entire group of participants:

- Computer with spreadsheet software
- Data file **Aunt Sarah and the Farm.xls**
- Computer projector and screen

Transparencies.

- *Aunt Sarah and the Farm*

Activity Sheet(s).

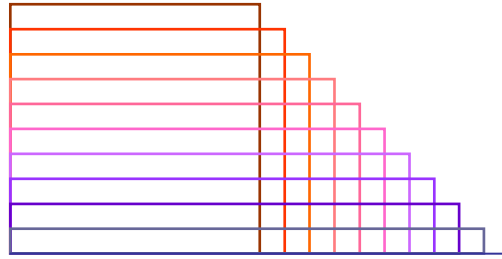
- *Aunt Sarah and the Farm*

Procedure:

Steps	Questions/Math Notes
<p>1. Read aloud the <i>Aunt Sarah and the Farm</i> two times. Allow participants to ask questions about the problem situation described.</p> <p>Ask participants to work in groups of 4 to begin exploring some of the mathematics of the situation.</p>	<p>To stimulate reflection, ask participants to predict the various changes in area and perimeters after the first three years have been computed.</p> <p>Ask:</p> <p><i>Is there a way to predict what happens in the 5th year? The 6th?</i></p> <p><i>How can we organize our work to make prediction easier? (Encourage a row and column layout).</i></p>
<p>2. Circulate among the groups as they work the problem. Encourage each group to complete their table of results until all ten years are in place.</p> <p>Ask participants to graphically represent the changes in area using graph paper.</p>	<p><i>To illustrate the linkage to functions encourage development of a single representation which shows all year's rather than ten separate diagrams.</i></p>
<p>3. If a group does not notice it after drawing the graphic representation point out that the areas from this situation can be modeled by the function $F(x)=(10+x)(10-x)$</p>	<p><i>How is your graphic reflected in the problem situation?</i></p>

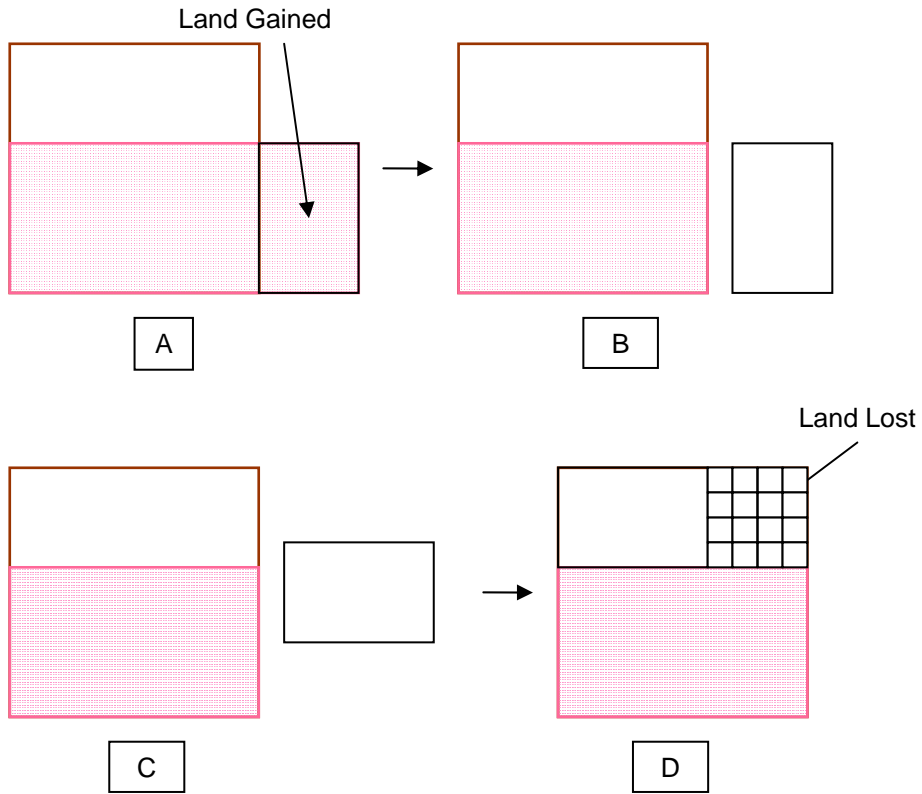
<p>where x is the number of years. Thus, a difference of squares is being generated.</p> <p>Select several small groups to present their findings and graphics.</p>	<p><i>Which portions of the functional representation are shown in your graphic?</i></p> <p><i>What about the differences and the sequences they generated? Where do we see them in the graphic?</i></p>
<p>4. Illustrate how to use a spreadsheet to model the situation and the graphs which can be generated. (See computer discussion which follows).</p>	<p><i>What additions to the story would we need to make to use a line graph? (Making the time between land adjustments approach zero).</i></p> <p><i>Does the shape of the graphs change as the time between adjustments decreases?</i></p>

Geometric Illustration of Aunt Sarah and the Farm Situation

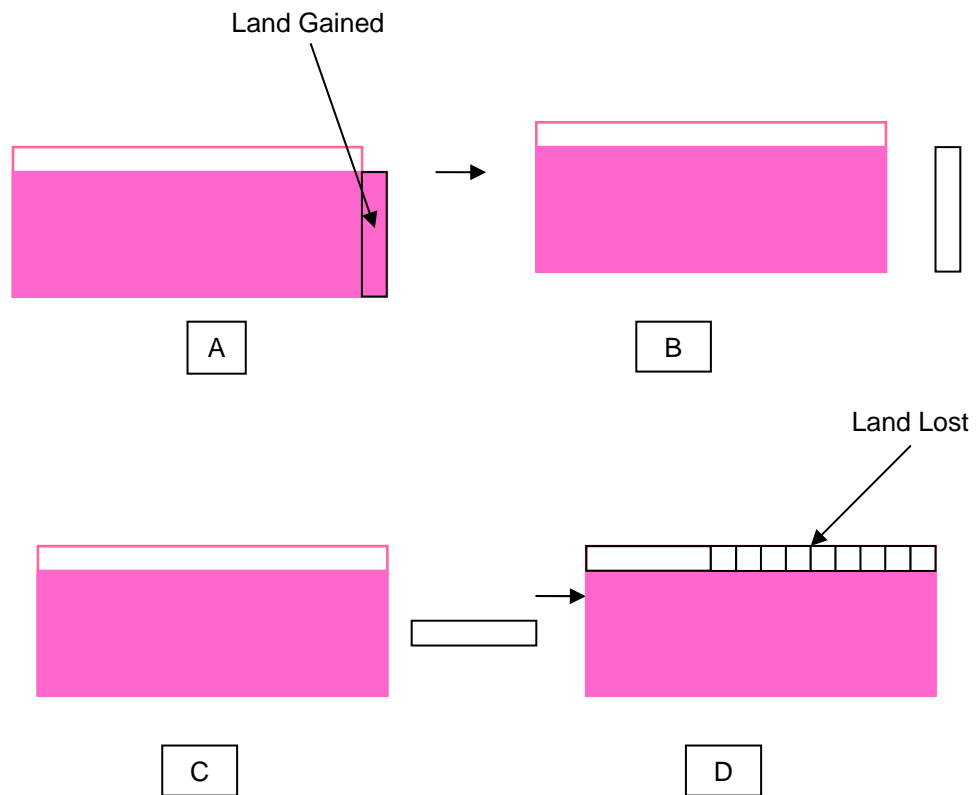


This representation can be used to explain where each of the columns in the spreadsheet comes from.

For example, to show the land lost for any given year relative to the beginning year (the fourth year is shown) take the rectangle of land gained for that year (A), rotate it (B and C), and place it inside the original figure to show the total amount lost (D).



To show the land lost for any given year relative to the preceding year (the difference between the fourth and fifth year is shown) take the rectangle of land gained for that year (A), rotate it (B and C), and place it inside the preceding figure to show the total amount lost (D).



Each of the other columns from the spreadsheet may be shown quite easily in a similar fashion.

Computer Discussion:

This is an ideal situation for computer modeling using a spreadsheet. The following screenshot from **Aunt Sarah and the Farm.xls** shows one possible way of representing the problem situation.¹

Elapsed Year	Length	Width	Area	Area loss from preceeding year	Area loss from first year	Perimeter (Fence Required)	Perimeter loss from preceeding year	Perimeter loss from first year
0	10	10	100			40		
1	11	9	99	1	1	40	0	0
2	12	8	96	3	4	40	0	0
3	13	7	91	5	9	40	0	0
4	14	6	84	7	16	40	0	0
5	15	5	75	9	25	40	0	0
6	16	4	64	11	36	40	0	0
7	17	3	51	13	49	40	0	0
8	18	2	36	15	64	40	0	0
9	19	1	19	17	81	40	0	0
10	20	0	0	19	100	40	0	0

The formula bar was used to create as many of the cells as possible. For example, cell B12 was defined as being:

B12 $f_x = 10 + A12$

In a like fashion each of the following cells were defined using the formula bar as being:

C12 $f_x = 10 - A12$

D12 $f_x = B12 * C12$

The increment in year was then defined in cell A13 as being:

A13 $f_x = A12 + 1$

This made it possible to copy cell A13 into cells A14 through A22. In a like fashion it is possible to copy cells B 12, C12 and D12 into cells B13 through B22, C13 through C22 and D13 through D22.

An examination of the formula bar for Columns C and D shows very clearly the difference of squares (i.e., the length and width) being used as a basis for the area calculation.

¹ All screenshots were created using Microsoft Excel.

The remaining columns look at some of the interesting interactions immediately springing from the problem situation.

Each cell in Column E, E13 for example, was computed using the following convention:

E13 $=D12-D13$

When this is done the sequence of odd numbers is generated, leading to questions concerning where this shows up in the graphical and functional representations generated in the group activity.

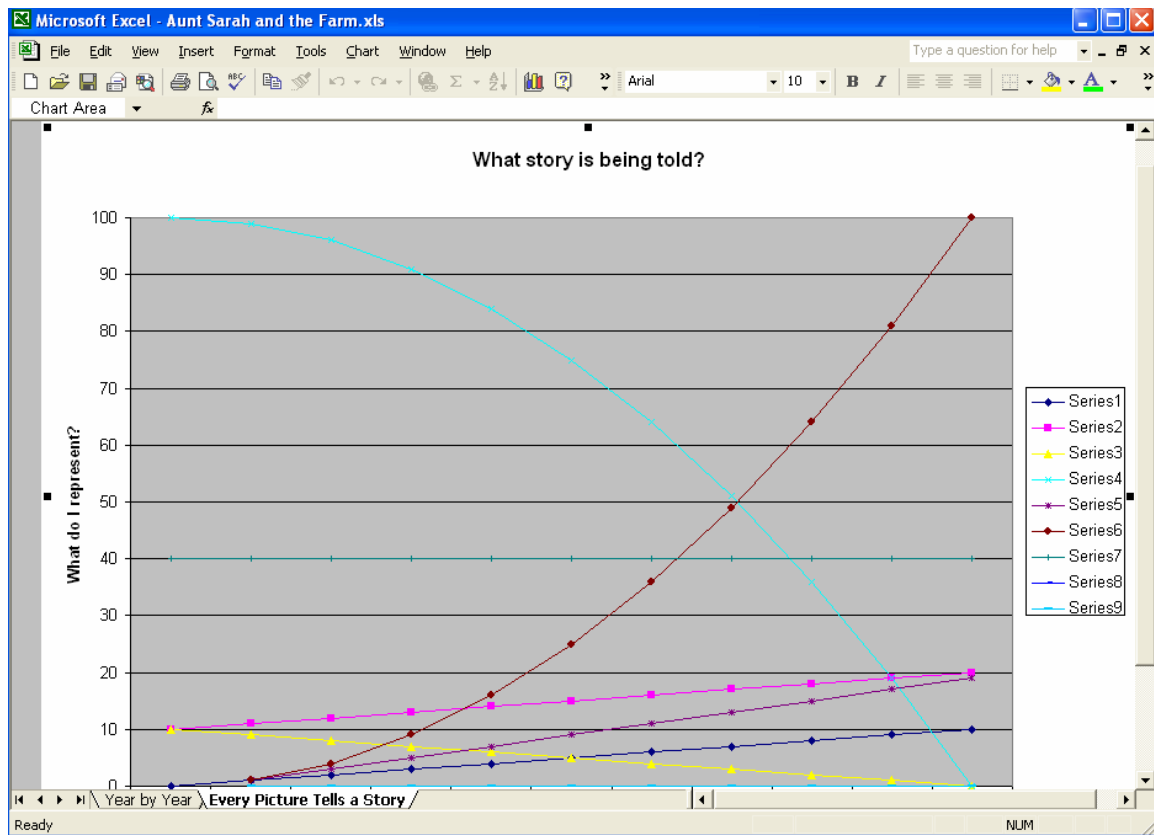
Column F was generated using the following:

F13 $=\$D\$12-D13$

The dollar sign preceding the D and the 12 indicates that this location will be *locked in* and used as the reference for each of the cells generated by copying it. This ensures that each subsequent years difference will be computed taking the first year as the comparison.

Now a sequence of squares is generated, once again leading to questions concerning where this shows up in the graphical and functional representations generated.

Using these columns the following graphs were generated:



It is now up to the participants to describe which series gives rise to each graph and why. They should also be able to link their graphic representation created on the graph paper to these graphs.


Extension:

Of course, in order for a line graph to be properly used a case must be made that there will not be any changes in the line as the difference between sampling times becomes infinitely small. This provides an easy link to the calculus which may be made via the spreadsheet.

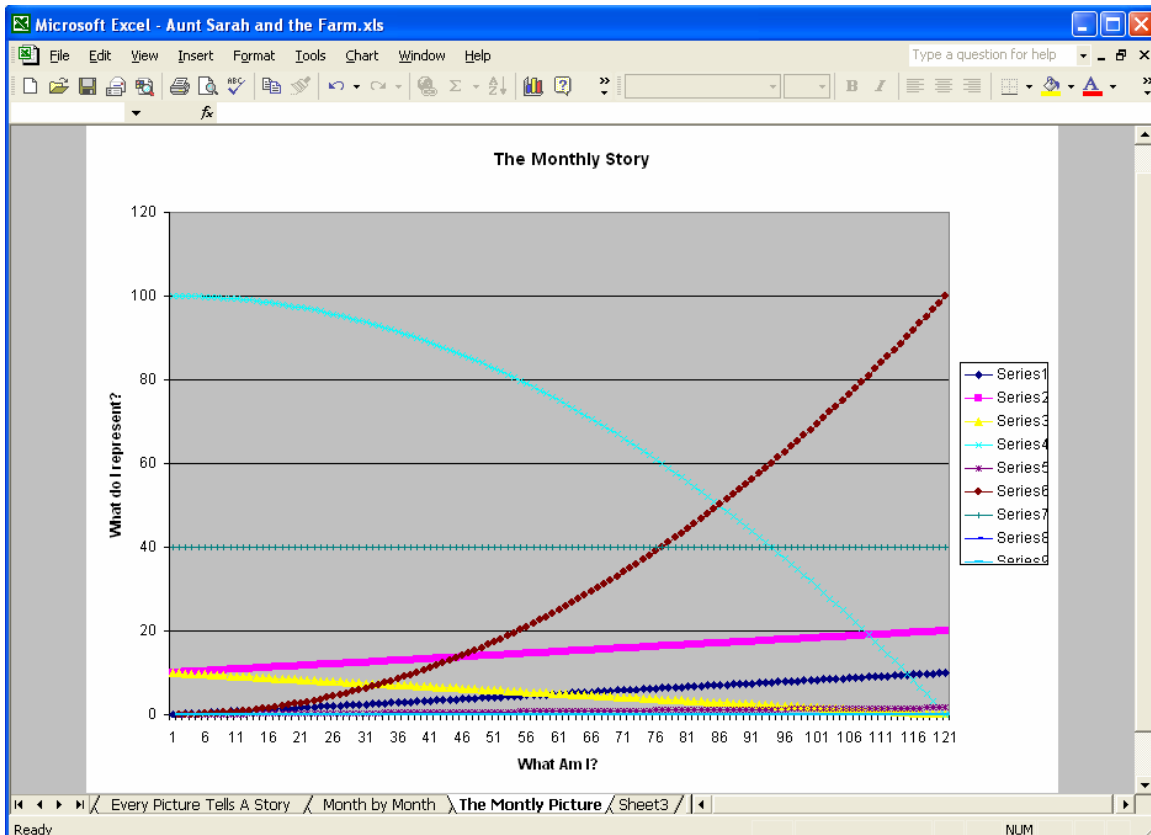
This can be shown by first changing the spreadsheet so that the “change point” occurs every month instead of every year. This action effectively changes the difference between points on the line graphs by 1/12. This is easily done by changing cell A13 to be:

A13 $=A12+1/12$

Now we can reconstruct the full table (all 120 rows of it!) with a simple set of copy instructions. The following shows the results of this action

	A	B	C	D	E	F	G	H	I	J	K
1	 <p style="text-align: center;">Aunt Sarah and the Farm</p>										
2											
3											
4											
5											
6											
7											
8											
9											
10											
11	Elapsed Time - by Months	Length	Width	Area	Area loss from preceeding Month	Area loss from first Month	Perimeter (Fence Required)	Perimeter loss from preceeding year	Perimeter loss from first year		
12	0	10	10	100			40				
13	0.08	10.08	9.92	99.99	0.01	0.01	40.00	0.00	0.00		
14	0.17	10.17	9.83	99.97	0.02	0.03	40.00	0.00	0.00		
15	0.25	10.25	9.75	99.94	0.03	0.06	40.00	0.00	0.00		
16	0.33	10.33	9.67	99.89	0.05	0.11	40.00	0.00	0.00		
17	0.42	10.42	9.58	99.83	0.06	0.17	40.00	0.00	0.00		
18	0.50	10.50	9.50	99.75	0.08	0.25	40.00	0.00	0.00		
19	0.58	10.58	9.42	99.66	0.09	0.34	40.00	0.00	0.00		
20	0.67	10.67	9.33	99.56	0.10	0.44	40.00	0.00	0.00		
21	0.75	10.75	9.25	99.44	0.12	0.56	40.00	0.00	0.00		
22	0.83	10.83	9.17	99.31	0.13	0.69	40.00	0.00	0.00		
23	0.92	10.92	9.08	99.16	0.15	0.84	40.00	0.00	0.00		
24	1.00	11.00	9.00	99.00	0.16	1.00	40.00	0.00	0.00		
25	1.08	11.08	8.92	98.83	0.17	1.17	40.00	0.00	0.00		
26	1.17	11.17	8.83	98.64	0.19	1.36	40.00	0.00	0.00		
27	1.25	11.25	8.75	98.44	0.20	1.56	40.00	0.00	0.00		

We can now reconstruct the earlier graphs using this more finely tuned set of measurements. When this is done the graphs created look like the following:



This is the identical shapes as shown in the earlier set of graphs. *The underlying equivalency can be better shown by changing the chart type to not plot the locations of the individual data points.*

In a like fashion we can narrow the limit to the day, the hour, the minute – to any degree we might choose... in each case since the underlying functions are the same the graphs will maintain the same shape!

References:

Blakeslee, D. W., & Chinn, W. G. (1975). The expected value of a random variable. *Introductory statistics and probability* (pp. 128-131). Boston, MA: Houghton Mifflin.