

6. Cardinality And The Strange Nature Of Infinity

There is no confusion about the size of sets that have finitely many elements. For example, everyone will agree that the sets

$$A = \{2, 1.5, 4/7, 0, -511\}$$

and

$$B = \{a, b, c, d, e, f, g\}$$

have a different number of elements. In fact, A has 5 elements and B has 7 elements. It is also clear that A is a smaller set than B . Interestingly, as we will see below, things are not nearly as predictable when sets contain infinitely many elements.

Definition 6.1: Let A be a set. If A is the empty set then the **cardinality** of A is 0. If A contains exactly n elements, for some $n \in \mathbb{N}$, then we say that the **cardinality** of A is n . In either case, we denote the cardinality of A by $|A|$, and we say that A is a **finite set** (because its cardinality is a finite number). Otherwise, we say that A is an **infinite set**.

Example 6.2: The set $A = \{2, 1.5, 4/7, 0, -511\}$ listed above is a finite set and the cardinality of A is 5; i.e. $|A| = 5$.

Example 6.3: The set of even natural numbers is an infinite set.

It seems natural to compare the sizes of infinite sets. In fact, there are a number of intuitive feelings that might lead us to make some (seemingly) obvious conclusions. For example, if we let E denote the set of even natural numbers, then E and \mathbb{N} are both infinite sets. However, it seems as though E has only half as many elements as \mathbb{N} .

$$\begin{aligned} E &= \{ 2, 4, 6, 8, 10, 12, \dots \} \\ \mathbb{N} &= \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots \} \end{aligned}$$

So, we might be tempted to say that $|E|$ is smaller than $|\mathbb{N}|$. It also seems as though the set \mathbb{N} is less than half the size of the set \mathbb{Z} .

$$\begin{aligned} \mathbb{N} &= \{ 1, 2, 3, 4, 5, \dots \} \\ \mathbb{Z} &= \{ \dots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots \} \end{aligned}$$

So we might be tempted to say that $|\mathbb{N}|$ is smaller than $|\mathbb{Z}|$. Finally, the size of \mathbb{Q} seems to completely dwarf the size of either \mathbb{Z} or \mathbb{N} , so we might be tempted to say that $|\mathbb{Q}|$ is

larger than both $|\mathbb{N}|$ and $|\mathbb{Z}|$. Unfortunately, all of these conclusions are incorrect! In addition, if you are thinking “Of course they are incorrect. Each of these sets is infinite, so their cardinality is the same.” then you are also wrong. In fact, it turns out that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$! There are also infinite sets which are much bigger than \mathbb{R} . For example, the set of real valued functions whose domains are \mathbb{R} and whose ranges are contained in \mathbb{R} is a much bigger set than \mathbb{R} . There are also other sets that are much bigger than this set. In fact, there are infinitely many degrees of infinity. We discuss only the smallest two below. Before we begin, we give a definition that seems to have nothing to do with counting the elements in a set.

Definition 6.4: Let A and B be sets and suppose f is a function with domain A and range contained in B . We say that f is a one-to-one (or 1-1) function if and only if

$$f(a) = f(b) \text{ implies } a = b.$$

Definition 6.5: Let A and B be infinite sets. We say that $|A| = |B|$ if and only if there is a one-to-one function f whose domain is A and whose range is B . We say that $|A| \leq |B|$ if and only if there is a one-to-one function f whose domain is A and whose range is contained in B . We say that $|A| < |B|$ if and only if $|A| \leq |B|$ and there does not exist a one-to-one function f whose domain is A and whose range is B .

Definition 6.6: Let A be a set. We say that A is a **countable set** if and only if A is a finite set or A is an infinite set and $|A| = |\mathbb{N}|$. We say A is a **countably infinite set** if and only if A is an infinite set which is countable. We say A is an **uncountable set** (or equivalently, an **uncountably infinite set**) if and only if A is not a countable set.

The next theorem tells us that \mathbb{N} is the smallest infinite set.

Theorem 6.7: If A is an infinite set then $|\mathbb{N}| \leq |A|$.

Proof: We know that the smallest infinite sets are countable ones. If A is an infinite set, then it is at least countable (i.e., countable or uncountable). Therefore, $|\mathbb{N}| \leq |A|$. ■

Remark 6.8: According to the theorem and definition above, the sets which are countably infinite are the smallest infinite sets. Notice that \mathbb{N} is certainly a countably infinite set since the identity function $f(n) = n$ is one-to-one from \mathbb{N} onto \mathbb{N} .

Theorem 6.9: Let A be an uncountable set. Then A is an infinite set and $|\mathbb{N}| < |A|$.

Proof: A is an uncountable set implies by definition that it is an infinite set.

As noted above, the sets which are countably infinite are the smallest infinite sets. Any set which is uncountably infinite has a bigger cardinality than the cardinality of \mathbb{N} . Hence, $|\mathbb{N}| < |A|$. ■

Example 6.10: The set of even natural numbers E is a countably infinite set. Consequently, $|E| = |\mathbb{N}|$.

Solution: We visualize E and \mathbb{N} as shown below. The arrows are inserted to help us construct a one-to-one function from \mathbb{N} to E .

$$\begin{array}{cccccccccccc}
 E & = & \{ & 2 & 4, & 6 & 8, & 10 & 12, & 14 & 16, & 18 & 20, & 22 & 24, & \dots & \} \\
 & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\
 \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & \dots & \}
 \end{array}$$

It appears as though the function $f(n) = 2n$ will give a one-to-one function with domain \mathbb{N} and range E . As a result, E is countably infinite and $|E| = |\mathbb{N}|$.

Example 6.11: The set \mathbb{Z} is a countably infinite set. Consequently, $|\mathbb{Z}| = |\mathbb{N}|$.

Solution: We visualize \mathbb{Z} and \mathbb{N} as shown below. The arrows are inserted to help us construct a one-to-one function from \mathbb{N} to \mathbb{Z} .

$$\begin{array}{cccccccccccc}
 \mathbb{N} & = & \{ & \dots & 11, & 9, & 7, & 5, & 3, & 1, & 2, & 4, & 6, & 8, & 10, & \dots & \} \\
 & & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\
 \mathbb{Z} & = & \{ & \dots & -5, & -4, & -3, & -2, & -1, & 0, & 1, & 2, & 3, & 4, & 5, & \dots & \}
 \end{array}$$

At first glance, the illustration above might not seem helpful. However, notice that the even natural numbers are all paired with an integer that is half of its value. Also, for the odd natural numbers, if we consider the ones that are greater than 3, subtract one from their value and then multiply by -1, we get the mapping shown above. We can summarize this by defining the function

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd and } n \geq 3 \end{cases}$$

You can verify that f corresponds to the arrows shown above, and that f is a one-to-one function with domain \mathbb{N} and range \mathbb{Z} . As a result, \mathbb{Z} is a countably infinite set and $|\mathbb{Z}| = |\mathbb{N}|$.

Theorem 6.12: Suppose A and B are infinite sets with $A \subseteq B$. If B is countably infinite, then A is countably infinite and $|A| = |B| = |\mathbb{N}|$.

Proof: Since A is an infinite set, it is either countably infinite or uncountably infinite. $A \subseteq B$ implies that B may have more elements than A , so the cardinality of A can not be bigger than the cardinality of B ; $|A| \leq |B|$. Since B is countably infinite, A can not be an uncountably infinite set. Hence, A is a countably infinite set; $|A| = |B| = |\mathbb{N}|$. ■

Theorem 6.13: Suppose A and B are sets and A is a countably infinite set. If there is a one-to-one function whose domain is A and whose range is B then B is countably infinite.

Proof: We know that $|A| = |B|$ since there is a one-to-one function f whose domain is A and whose range is B (see Definition 6.5). Hence, A is countably infinite implies that B is also a countably infinite set. ■

Example 6.14: The set \mathbb{Q} is a countably infinite set. That is, $|\mathbb{Q}| = |\mathbb{N}|$.

Solution: Notice that there are a lot of representations for any given rational number. For example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$. However, there is one reduced form for a rational number. We say $q \in \mathbb{Q}$ is **written in reduced form** if and only if either q is written as an integer or q is written in the form a/b where $a, b \in \mathbb{Z}$ with $b \geq 2$ and $\gcd(a, b) = 1$. We will use the reduced form of a rational number to create a one-to-one function from \mathbb{Q} to an infinite subset of \mathbb{Z} . Once we do this, from theorem 6.12 above, this subset must be countably infinite, and consequently it will follow from theorem 6.13 that \mathbb{Q} is countably infinite. Our strategy is to use the idea of counting in base 11. To this end, let \odot be the single digit representation for 10 in base 11. Now, define

$$f(q) = \begin{cases} q_{11} & \text{if } q \text{ is an integer} \\ a \odot b_{11} & \text{if } q \text{ is not an integer and } a/b \text{ is the reduced form of } q \end{cases}$$

For example,

$$f(12) = 12_{11} = 1(11) + 2 = 13$$

and

$$f(3/5) = 3 \odot 5_{11} = (3(11)^2 + 10(11) + 5) = 478$$

You can check that f is a one-to-one function with domain \mathbb{Q} and range contained in \mathbb{Z} . Also, the range of f is certainly infinite, so the range of f is countably infinite by theorem 6.12. Therefore, from theorem 6.13, \mathbb{Q} is countably infinite and $|\mathbb{Q}| = |\mathbb{N}|$.

Theorem 6.15: If A and B are countable sets, then $A \cup B$ is a countable set.

Proof: If A and B are finite sets then $|A \cup B| = |A| + |B| - |A \cap B|$. So, $A \cup B$ is a countable set.

Suppose A and B are countably infinite sets. Let $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$. Then $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$. If $a_i = b_j$ for some i and j , then such elements are written just once, which does not change the cardinality when there are infinitely many elements (the worst case is $a_i = b_j$ for all i and j , and in that case

$A \cup B = \{a_1, a_2, a_3, \dots\}$ which is a countable set). Actually, being able to write $A \cup B$ as $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$ shows that it is a countably infinite set.

But it is better to say that there is a 1-1 and onto function from \mathbb{N} to $A \cup B$. Let $A \cup B = \{c_1, c_2, c_3, \dots\}$ be our set after writing the same elements once. Then, $f(n) = c_n$ is a 1-1 and onto function. Hence, $|A \cup B| = |\mathbb{N}|$. ■

Definition 6.16: The set of **irrational numbers** is the set of real numbers which are not rational numbers.

Theorem 6.17: Both \mathbb{R} and the set of irrational numbers are uncountable sets.

Proof: We start by showing that \mathbb{R} is an uncountable set.

It is better to deal with a smaller (relatively!) set; let's take the interval $(0,1)$.

Any number in this interval can be expressed as a unique, never-ending decimal.

If there were countably many numbers in this interval, we could list them in a table:

| | |
|-------------------------------|------------------------------------|
| 1st number: | $0, x_1^1 x_2^1 x_3^1 x_4^1 \dots$ |
| 2nd number: | $0, x_1^2 x_2^2 x_3^2 x_4^2 \dots$ |
| 3rd number: | $0, x_1^3 x_2^3 x_3^3 x_4^3 \dots$ |
| 4th number: | $0, x_1^4 x_2^4 x_3^4 x_4^4 \dots$ |
| | |

Here, each x_i^j denotes a digit between 0 and 9.

Now, let $x = 0, x_1 x_2 x_3 x_4 \dots$ where

- $x_1 = 0$ if $x_1^1 \geq 1$ and $x_1 = 1$ if $x_1^1 = 0$
- $x_2 = 0$ if $x_2^2 \geq 1$ and $x_2 = 1$ if $x_2^2 = 0$
- $x_3 = 0$ if $x_3^3 \geq 1$ and $x_3 = 1$ if $x_3^3 = 0$
-

This element x is different from the first number in our list since they differ in the first decimal digit (first entry), x is different from the second number in the list since they differ in the second decimal digit (second entry), etc. Thus, x can not be in the list since it differs with the n^{th} element in the list in the n^{th} entry. Hence, x is a number in the interval $(0,1)$ but it is not in the list. That is, we are unable to list all numbers in $(0,1)$; $(0,1)$ is an uncountable set.

Since \mathbb{R} is the union of lots of such uncountable intervals, we can conclude that \mathbb{R} is uncountable.

We know that \mathbb{R} is the union of the set of rational numbers and the set of irrational numbers. If the set of irrational numbers were countable, then since the union of two countable sets is countable (Theorem 6.15), \mathbb{R} would be countable. But we just showed that it is not; hence, the set of irrational numbers is an uncountable set. ■

Exercises

1. Let T be the set of natural numbers which are multiples of 3. Show that T is a countably infinite set.
2. Show that the set of prime numbers is a countably infinite set.
3. Show that the set of whole numbers is a countably infinite set.
4. Show that if a is a rational number and b is an irrational number then $a + b$ is an irrational number.
5. Is the set of irrational numbers closed under addition?
6. Is the set of irrational numbers closed under multiplication?
7. Is the product of an irrational number and a rational number an irrational number? What if the rational number is nonzero?
8. Another idea for “counting” the positive rational numbers is given as follows. First, create an infinite table (or matrix) as shown below.

$$\begin{pmatrix} 1/1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ 2/1 & 2/2 & 2/3 & 2/4 & 2/5 & 2/6 & 2/7 & \dots \\ 3/1 & 3/2 & 3/3 & 3/4 & 3/5 & 3/6 & 3/7 & \dots \\ 4/1 & 4/2 & 4/3 & 4/4 & 4/5 & 4/6 & 4/7 & \dots \\ 5/1 & 5/2 & 5/3 & 5/4 & 5/5 & 5/6 & 5/7 & \dots \\ 6/1 & 6/2 & 6/3 & 6/4 & 6/5 & 6/6 & 6/7 & \dots \\ 7/1 & 7/2 & 7/3 & 7/4 & 7/5 & 7/6 & 7/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Notice that the rational numbers in this table are not in reduced form. However, every positive rational number is easy to find in this table. If $m, n \in \mathbb{N}$ then m/n can be found in row m and column n . Now, move through this matrix starting in the upper left hand corner, following the natural numbers in the matrix below.

| | | | | | | | |
|----|-------------|----|----|----|----|----|-----|
| 1 | 3 | 6 | 10 | 15 | 21 | 28 | ... |
| 2 | 5 | 9 | 14 | 20 | 27 | | ... |
| 4 | 8 | 13 | 19 | 26 | | | ... |
| 7 | 12 | 18 | 25 | | | | ... |
| 11 | 17 | 24 | | | | | ... |
| 16 | 23 | | | | | | ... |
| 22 | <i>etc.</i> | | | | | | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

What natural number will correspond to 11/13? Can you give the natural number that corresponds to m/n when $m, n \in \mathbb{N}$? Explain how to use the ideas above to show that the set of positive rational numbers is countably infinite.

Solutions:

1. Let T be the set of natural numbers which are multiples of 3. So, $T = \{3n \mid n \in \mathbb{N}\}$.

We need to find a 1-1 function from \mathbb{N} to T .

Define $f(n) = 3n$. Then the domain of this function is \mathbb{N} , and the range is T . Is it one-to-one?

$f(m) = f(n)$ implies $3m = 3n$, which means $m = n$. So, yes, it is one-to-one.

Hence, T is countably infinite and $|T| = |\mathbb{N}|$.

2. Let $P = \{p \mid p \in \mathbb{N} \text{ and } p \text{ is prime}\}$. We know that there is no largest prime number, i.e. P has infinitely many elements. So, by Theorem 6.6, $|\mathbb{N}| \leq |P|$.

Let us define $f(p) = p$ on P . Then f is a 1-1 function whose domain is P and range is P . We know that $P \subset \mathbb{N}$. Hence, f is a function whose domain is P and whose range is contained in \mathbb{N} ; by the definition 6.5, $|P| \leq |\mathbb{N}|$.

Therefore, we can conclude that $|\mathbb{N}| = |P|$; that is, P is a countably infinite set.

3. $\mathbb{Z}_{0,+} = \{0, 1, 2, 3, 4, \dots\}$. Let us define a function on $\mathbb{Z}_{0,+}$ such that $f(n) = n + 1$. Then f is clearly 1-1. The domain of f is $\mathbb{Z}_{0,+}$, the range of f is \mathbb{N} . Hence, $\mathbb{Z}_{0,+}$ is a countably infinite set; $|\mathbb{N}| = |\mathbb{Z}_{0,+}|$.

4. Let a be a rational number and b be an irrational number.

It will be easier to find the natural number corresponding to 11/13 if we derive a formula first.

Notice that ;

$$m = 1; 1 + 2 = 3, 1 + 2 + 3 = 6, 1 + 2 + 3 + 4 = 10, 1 + 2 + 3 + 4 + 5 = 15, \dots$$

$m = 2; 2 + 3 = 5, 2 + 3 + 4 = 9, 2 + 3 + 4 + 5 = 14, 2 + 3 + 4 + 5 + 6 = 20, \dots$, and so on.. So, there is a pattern in the way the entries increase.

By remembering that $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$ and doing some calculations , we can derive a formula for finding the entry in the m^{th} row and n^{th} column; $A_{m,n}$.

$$A_{m,n} = \frac{(m+n-1)(m+n)}{2} - (m-1).$$

The natural number corresponding to 11/13 is in the 11th row and 13th column. Hence,

$$A_{11,13} = \frac{23 \cdot 24}{2} - 10 = 266.$$

By deriving a formula to the entry $A_{m,n}$, we found a 1-1 function whose domain

is \mathbb{Q} and range is \mathbb{N} ; specifically, $f\left(\frac{m}{n}\right) = \frac{(m+n-1)(m+n)}{2} - (n-1)$. Hence,

\mathbb{Q} is a countably infinite set.