5. Integers, Whole Numbers And Rational Numbers

The set of natural numbers, $\mathbb{N}$, is the building block for most of the real number system. But $\mathbb{N}$ is inadequate for measuring and describing physical quantities, it does not contain the additive identity 0, and it is not closed under subtraction and division. In this section, we use $\mathbb{N}$ to create the set of integers, the set of whole numbers and the set of rational numbers. The latter set resolves most of the shortcomings of $\mathbb{N}$.

We begin by giving a definition for the set of integers and a special subset referred to as the whole numbers. Then we extend the ideas of divisor, multiple, prime factorization, greatest common divisor and least common multiple to numbers in these sets.

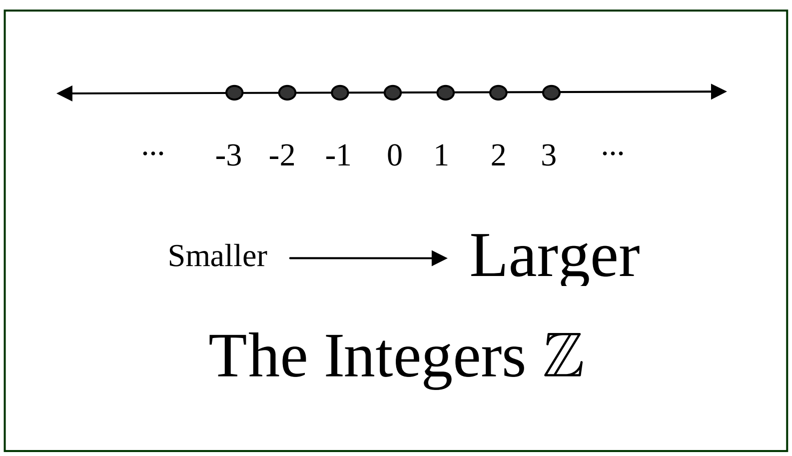
Definition 5.1: The set of **integers**, $\mathbb{Z}$, is the subset of $\mathbb{R}$ given by

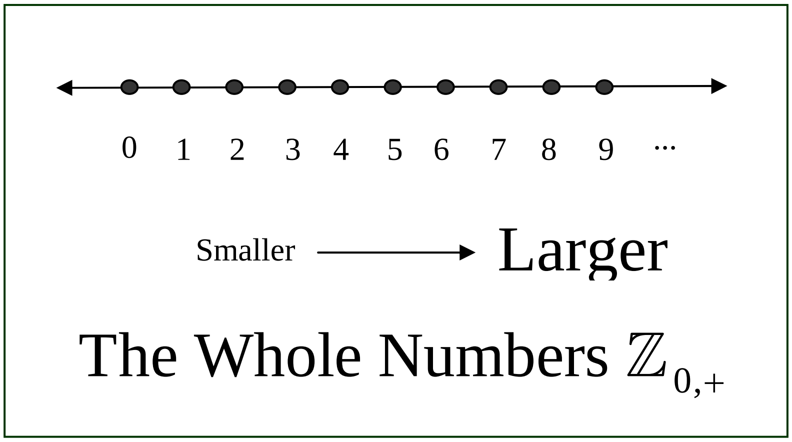
$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

The set of **whole numbers**, $\mathbb{Z}_{0,+}$, is the subset of $\mathbb{Z}$ given by

$$\mathbb{Z}_{0,+} = \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

These sets can be visualized within the real number line as shown below.





Notice that both $\mathbb{Z}$ and $\mathbb{Z}_{0,+}$ contain both the additive identity 0 and the multiplicative identity 1. You will be asked in the exercises to verify the following:

Z1 The set of whole numbers, $\mathbb{Z}_{0,+}$, is closed under both addition and multiplication.

Z2 The set of whole numbers is not closed under subtraction or division.

Z3 The set of integers, $\mathbb{Z}$, is closed under addition, subtraction and multiplication.

Z4 The set of integers, $\mathbb{Z}$, is not closed under division.

From the standpoint of order, the integers $\mathbb{Z}$ are much different from the natural numbers $\mathbb{N}$. There is a smallest element in $\mathbb{N}$; namely 1. There is no smallest element in $\mathbb{Z}$. Notice that the set of whole numbers $\mathbb{Z}_{0,+}$ does have a smallest element; namely 0. None of the sets $\mathbb{N}$, $\mathbb{Z}_{0,+}$ or $\mathbb{Z}$ contains a largest element.

From an algebraic standpoint, the set of integers $\mathbb{Z}$ is not so different from the set of natural numbers $\mathbb{N}$. The new additional entries in $\mathbb{Z}$ are 0 and the additive inverses of the entries in $\mathbb{N}$. As a result, we can extend the ideas of divisors, multiples and prime factorizations to the nonzero elements in $\mathbb{Z}$.

Definition 5.2: Let $m,n \in \mathbb{Z}$. We say that ***m* is a multiple of *n*** if and only if there is an integer *k* so that $m = kn$. We say that ***n* is a divisor of *m*** if and only if *m* is a multiple of *n*.

Theorem 5.3: Let $m,n \in \mathbb{Z}$ so that ***m* is a multiple of *n***. Then $-m$ is a multiple of *n* and $-n$ is a divisor of *m*.

Proof: Let $m, n \in \mathbb{Z}$. We say that ***m* is a multiple of *n*** if and only if there is an integer k so that $m = kn$. k is an integer means $-k$ is also an integer.

$m = kn \;\Rightarrow\; m = (-k)(-n)$. Hence m is a multiple of $-n$.
Then, by definition, $-n$ is a divisor of m. ■

Remark 5.4: Both -1 and 1 are divisors of every integer, 0 is a multiple of every integer, and every integer is a divisor of 0.

Definition 5.5: Let $a, b \in \mathbb{Z}$ with $a, b \neq 0$. The **greatest common divisor of *a* and *b***, denoted $\gcd(a,b)$, is the largest integer that divides both a and b. The **least common multiple of *a* and *b***, denoted $\operatorname{lcm}(a,b)$, is the smallest <u>positive</u> multiple of both a and b.

Remark 5.6: We require that $a, b \neq 0$ in the definition above since every number is a divisor of 0. As a result there is no greatest common divisor of 0 and itself. In addition, there is no positive multiple of 0, so the least common multiple of 0 and b can not be defined for any integer b.

The result below allows us to use earlier results to find $\gcd(a,b)$ and $\operatorname{lcm}(a,b)$ when $a, b \in \mathbb{Z}$ with $a, b \neq 0$.

Theorem 5.7: Let $a, b \in \mathbb{Z}$ with $a, b \neq 0$. Then

$$\gcd(a,b) = \gcd(-a,b) = \gcd(a,-b) = \gcd(-a,-b)$$

and

$$\operatorname{lcm}(a,b) = \operatorname{lcm}(-a,b) = \operatorname{lcm}(a,-b) = \operatorname{lcm}(-a,-b)$$

Proof: Let $c = \gcd(a,b)$. Then c is the largest integer that divides both a and b. Any integer that divides a and b also divides $-a$ and $-b$ (see Theorem-5.3), and vice versa. Hence, c is the largest integer that divides both a and b means c is also the largest integer that divides both $-a$ and b. Same argument holds for the other cases. Hence,
$\gcd(a,b) = \gcd(-a,b) = \gcd(a,-b) = \gcd(-a,-b)$.

Similarly, any integer that is a multiple of a and b is also a multiple of $-a$ and $-b$ (see Theorem-5.3), and vice versa. Hence, the smallest positive integer that is a multiple of both a and b is same as the smallest positive integer that is a multiple of $-a$ and $-b$. Hence,

$$\operatorname{lcm}(a,b) = \operatorname{lcm}(-a,b) = \operatorname{lcm}(a,-b) = \operatorname{lcm}(-a,-b). \blacksquare$$

Example 5.8: Find $\gcd(-214,360)$ and $\operatorname{lcm}(-214,360)$.

Solution: From the theorem above and an earlier result,

$$\gcd(-214,360) = \gcd(214,360)$$

and

$$\gcd(214,360)\operatorname{lcm}(214,360) = (214)(360) = 77040$$

We can use either prime factorization or the division algorithm to show $\gcd(214,360) = 2$. Consequently, $\operatorname{lcm}(214,360) = 38520$. Therefore,

$$\gcd(-214,360) = 2 \text{ and } \operatorname{lcm}(-214,360) = 38520.$$

We are now ready to discuss the set of rational numbers.

Definition 5.9: The set of rational numbers, $\mathbb{Q}$, is given by

$$\mathbb{Q} = \left\{ a/b \;\middle|\; a,b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

Notice that every integer *m* is a rational number since $m = m/1$. Also, the set of natural numbers, the set of whole numbers, the set of integers and the set of rational numbers are ordered by set inclusion as shown below.

$$\mathbb{N} \subseteq \mathbb{Z}_{0,+} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

The set of rational numbers $\mathbb{Q}$ has much more algebraic structure than any of $\mathbb{N}$, $\mathbb{Z}_{0,+}$ or $\mathbb{Z}$. The proof of the following theorem is relegated to the exercises.

Theorem 5.10: The operations $+$ and $\times$ satisfy the field axioms on $\mathbb{Q}$. That is, for every $a,b,c \in \mathbb{Q}$,

Name	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a \times b = b \times a$
Associative Property	$(a+b)+c = a+(b+c)$	$(a \times b) \times c = a \times (b \times c)$
Distributive Property	$a \times (b+c) = a \times b + a \times c$	
Identity	$a+0=0+a=a$	$a \times 1 = 1 \times a = a$
Inverse	There exists $-a \in \mathbb{Q}$ so that $a+(-a)=(-a)+a=0$	if $a \neq 0$ there exists $a^{-1} \in \mathbb{Q}$ so that $a \times a^{-1} = a^{-1} \times a = 1$

The order properties of $\mathbb{Q}$ is much more complicated than the order properties of $\mathbb{Z}$. We know that the elements of $\mathbb{Z}$ are ordered as follows:

$$\cdots < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$$

There is no such ordering for $\mathbb{Q}$. Also, it is not always easy to determine (by inspection) whether one rational number is smaller or larger than another. For example, it is not immediately obvious that

$$\frac{101123}{79734} < \frac{42136}{33165}$$

The following result can help determine the ordering of two rational numbers.

Theorem 5.11: Let $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$.

- $a/b > 0$ if and only if $ab > 0$.
- $a/b < 0$ if and only if $ab < 0$.
- If $a/b < 0$ and $c/d > 0$ then $a/b < c/d$.
- If $a/b > 0$ and $c/d > 0$ then $a/b < c/d$ if and only if $ad < bc$.
- If $a/b < 0$ and $c/d < 0$ then $a/b < c/d$ if and only if $ad > bc$.
- $a/b = c/d$ if and only if $ad = bc$.

Proof:

- $a/b > 0$ implies that a and b have the same signs, i.e. they are both positive or both negative. And this implies that $ab > 0$.
  Moreover, $ab > 0$ implies that a and b have the same signs, which means $a/b > 0$.
  Hence, $a/b > 0$ if and only if $ab > 0$.

- $a/b < 0$ means a and b have the opposite signs, i.e. one of them is positive and the other is negative. Therefore, $ab < 0$. Similarly, $ab < 0$ implies that $a/b < 0$.
- Any negative number is less than any positive number. Hence, If $a/b < 0$ and $c/d > 0$ then $a/b < c/d$.
- Suppose $a/b > 0$ and $c/d > 0$. If $a/b < c/d$, then multiply both sides of the inequality with d/c (which is positive!); $\frac{a}{b} \cdot \frac{d}{c} < \frac{c}{d} \cdot \frac{d}{c} = 1$. Hence, $\frac{ad}{bc} < 1$. This implies that $ad < bc$.
  If $ad < bc$, then $\frac{ad}{bc} < 1$. Since $\frac{ad}{bc} \cdot \frac{c}{d} < 1 \cdot \frac{c}{d}$, we get $a/b < c/d$.

- Suppose $a/b < 0$ and $c/d < 0$. If $a/b < c/d$, then $\frac{a}{b} \cdot \frac{d}{c} > \frac{c}{d} \cdot \frac{d}{c} = 1$ (since we multiply the inequality with a negative number, "<" becomes ">"). So, $\frac{ad}{bc} > 1$. This implies that $ad > bc$.
  If $ad > bc$, then $\frac{ad}{bc} > 1$. $\frac{ad}{bc} \cdot \frac{c}{d} < 1 \cdot \frac{c}{d}$ implies that $a/b < c/d$.

- If $a/b = c/d$, then $\frac{a}{b} \cdot \frac{d}{c} = \frac{c}{d} \cdot \frac{d}{c} = 1$. That is, $\frac{ad}{bc} = 1$. Hence, $ad = bc$.

If $ad = bc$, then $\frac{ad}{bc} = 1$. That is, $\frac{a}{b} \cdot \frac{d}{c} = 1$. So $\frac{a}{b} \cdot \frac{d}{c} \cdot \frac{c}{d} = 1 \cdot \frac{c}{d}$ Hence, $a/b = c/d$. ■

Example 5.12: Show that $\frac{101123}{79734} < \frac{42136}{33165}$.

Solution: The rational numbers $\frac{101123}{79734}$ and $\frac{42136}{33165}$ are both positive. Also,

$$(101123)(33165) = 3353744295$$

and

$$(79734)(42136) = 3359671824.$$

Since $3359671824 > 3353744295$, we can see from property 4 of the theorem above that

$$\frac{101123}{79734} < \frac{42136}{33165}.$$

You should be aware that it is possible to use long division to write any rational number in decimal form. For example,

$$\frac{1}{2} = 0.5, \ \frac{1}{4} = 0.25, \ \frac{1}{3} = 0.\bar{3}, \ \frac{4}{13} = 0.\overline{307692} \text{ and } \frac{53}{130} = 0.407692\overline{307692}$$

where the *overbar* in the last three decimal numbers indicates that the digits repeat indefinitely. That is,

$$\frac{1}{3} = 0.33333333333333\cdots$$

and

$$\frac{4}{13} = 0.\underbrace{307692}\underbrace{307692}\underbrace{307692}\underbrace{307692}\cdots$$

Notice that the decimal representations for the five rational numbers above either terminate (end) or eventually repeat indefinitely. The result below states that this is a defining characteristic of rational numbers.

Theorem 5.13: A number $r \in \mathbb{R}$ is a rational number if and only if *r* has a decimal representation that either terminates or eventually repeats indefinitely.

Unfortunately, if a rational number has a decimal representation that terminates, then the number also has another decimal representation. This can be demonstrated with some simple arithmetic. Note that 1/9 is given by the repeating decimal

$$\frac{1}{9} = .1111111111\cdots$$

Also, $9\left(\frac{1}{9}\right) = 1$. So, $9\left(.1111111111\cdots\right) = 1$. But we can see from long multiplication that

$$\begin{array}{r} .1111111111\cdots \\ \times 9 \\ \hline .9999999999\cdots \end{array}$$

Consequently, $.\overline{9} = 1$. Similarly,

$$.0\overline{9} = .1,\ .00\overline{9} = .01,\ .000\overline{9} = .001, \text{ etc.}$$

As a result,

$$.25 = .24\overline{9} \text{ and } .2 = .1\overline{9}$$

It is also much more complicated to visualize the rational numbers within the real number line than it is to visualize the integers within the real number line. This is because the rational numbers are nearly everywhere on the real line! We make this idea precise in the definition and theorem given below.

Definition 5.14: A subset *S* of $\mathbb{R}$ is said to be dense in $\mathbb{R}$ if and only if for every $a,b \in \mathbb{R}$ with $a < b$ there is an element $c \in S$ so that $a < c < b$.

Remark 5.15: In lay terms, the statement above says that a dense subset of $\mathbb{R}$ is one that can be used to give arbitrarily close approximations of every number in $\mathbb{R}$. We can see this by imagining that the number *a* referred to in the definition is the real number we wish to approximate and *b* is a real number which is only slightly larger than *a*. Then the requirement that *c* lies between *a* and *b* implies that *c* is approximately *a*. The closeness of this approximation depends upon how close *b* is chosen to *a*.

Theorem 5.16: $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

At one time, it was thought that all real numbers were rational numbers. In fact, one story (which might be folk lore) states that the philosopher Hippasus demonstrated that $\sqrt{2}$ was not a rational number while he was at sea with some of his colleagues. When he told them of his discovery, they became enraged and threw him overboard. We will see shortly that Hippasus was right!

Exercises:

1. Verify properties **Z1 – Z4**.
2. Let $m,a,b \in \mathbb{Z}$ with $a,b \neq 0$ and $m > 0$. Show that $\gcd(ma,mb) = m\gcd(a,b)$; i.e. the greatest common divisor satisfies a distributive property.
3. Let $a,b,c \in \mathbb{Z}$ with $a,b,c \neq 0$. Show that $\gcd(\gcd(a,b),c) = \gcd(a,\gcd(b,c))$; i.e. the greatest common divisor satisfies an associative property.
4. Let $a,b \in \mathbb{Z}$ with $a,b \neq 0$. Show that $\gcd(a,b) = \gcd(b,a)$; i.e. the greatest common divisor satisfies a commutative property.

5. Let $a,b \in \mathbb{N}$ with $a,b \neq 0$. Show that $\operatorname{lcm}(a,\gcd(a,b)) = a$ and $\gcd(a,\operatorname{lcm}(a,b)) = a$.
6. Let $a \in \mathbb{N}$ with $a \neq 0$. Show that $\operatorname{lcm}(a,a) = a$; i.e. the least common multiple is idempotent.
7. Let $a,b \in \mathbb{Z}$ with $a,b \neq 0$. Show that $\operatorname{lcm}(a,b) = \operatorname{lcm}\left(b,a\right)$; i.e. the least common multiple satisfies a commutative property.
8. Let $a,b,c \in \mathbb{Z}$ with $a,b,c \neq 0$. Show that $\operatorname{lcm}\left(\operatorname{lcm}(a,b),c\right) = \operatorname{lcm}\left(a,\operatorname{lcm}(b,c)\right)$; i.e. the least common multiple satisfies an associative property.
9. Let $m,a,b \in \mathbb{Z}$ with $a,b \neq 0,\ m > 0$. Show that $\operatorname{lcm}\left(ma,mb\right) = m\operatorname{lcm}\left(a,b\right)$; i.e. the least common multiple satisfies a distributive property.
10. A colleague claims the probability is 1/3 that a randomly chosen rational number (in reduced form) has an even denominator. Does this statement make any sense?
11. Show that you can always find another rational number between any two distinct rational numbers.
12. Prove Theorem 5.10 (i.e. show that $\mathbb{Q}$ satisfies all of the field axioms).
13. Although decimal representations for rational numbers must either terminate or eventually repeat indefinitely, the length of the repeating portion of a decimal representation can be very long and complicated. For example, the number $13.4678913456178\overline{0101343719842112}$ is a rational number. Use long division to determine the repeating decimal expansion for 2/29.

Solutions:

1. **Z1**: The set of whole numbers, $\mathbb{Z}_{0,+}$, is closed under both addition and multiplication.

Let $m,n \in \mathbb{Z}_{0,+}$. If m or n is 0, then $mn = 0 \in \mathbb{Z}_{0,+}$. Suppose neither m nor n is zero. Then $m,n \in \mathbb{N}$. We know that $\mathbb{N}$ is closed under multiplication, so $mn \in \mathbb{N}$ and $\mathbb{N} \subseteq \mathbb{Z}_{0,+}$; $mn \in \mathbb{Z}_{0,+}$.

Hence $\mathbb{Z}_{0,+}$ is closed under multiplication.

If m and n are both 0, then $m+n = 0 \in \mathbb{Z}_{0,+}$. If m is 0 and n is not, then $m+n = n \in \mathbb{Z}_{0,+}$.

Suppose neither m nor n is zero. Then $m,n \in \mathbb{N}$. We know that $\mathbb{N}$ is closed under addition, so $m+n \in \mathbb{N}$ and $\mathbb{N} \subseteq \mathbb{Z}_{0,+}$; $m+n \in \mathbb{Z}_{0,+}$.

Hence $\mathbb{Z}_{0,+}$ is closed under addition.

Z2: The set of whole numbers is not closed under subtraction or division.

It is enough to give an example to show that $\mathbb{Z}_{0,+}$ is not closed under subtraction or division.

$2, 5 \in \mathbb{Z}_{0,+}$, however $2-5=-3 \notin \mathbb{Z}_{0,+}$ and $\dfrac{2}{5}=0.4 \notin \mathbb{Z}_{0,+}$.

Z3: The set of integers, $\mathbb{Z}$, is closed under addition, subtraction and multiplication.

Let $m, n \in \mathbb{Z}$. If $m, n \in \mathbb{Z}_{0,+}$, then $m+n \in \mathbb{Z}_{0,+} \subseteq \mathbb{Z}$; $m+n \in \mathbb{Z}$.
If both $m, n < 0$, then $(-m),(-n) \in \mathbb{Z}_{0,+}$, and hence $(-m)+(-n) \in \mathbb{Z}_{0,+}$; i.e. $-(m+n) \in \mathbb{Z}_{0,+}$. So $m+n \in \mathbb{Z}$.
Suppose $m<0$ and $n \in \mathbb{Z}_{0,+}$. If $-m>n$, then $(-m)-n \in \mathbb{Z}_{0,+}$, i.e. $-(m+n) \in \mathbb{Z}_{0,+}$, hence $m+n \in \mathbb{Z}$. If $-m<n$, then $m+n \in \mathbb{Z}_{0,+}$, hence $m+n \in \mathbb{Z}$.
Thus, $\mathbb{Z}$ is closed under addition.

Let $m, n \in \mathbb{Z}$. Then $m,(-n) \in \mathbb{Z}$ and $m+(-n)=m-n \in \mathbb{Z}$ (since $\mathbb{Z}$ is closed under addition).
Hence, $\mathbb{Z}$ is closed under subtraction.

Let $m, n \in \mathbb{Z}$. If $m, n \in \mathbb{Z}_{0,+}$, then $m \cdot n \in \mathbb{Z}_{0,+} \subseteq \mathbb{Z}$; $m \cdot n \in \mathbb{Z}$.
If both $m, n < 0$, then $(-m),(-n) \in \mathbb{Z}_{0,+}$, and hence $(-m)\cdot(-n) \in \mathbb{Z}_{0,+}$; i.e. $m \cdot n \in \mathbb{Z}_{0,+}$. So $m \cdot n \in \mathbb{Z}$.
Suppose $m<0$ and $n \in \mathbb{Z}_{0,+}$. Then $-m \in \mathbb{Z}_{0,+}$ and hence $(-m)\cdot n \in \mathbb{Z}_{0,+}$, i.e. $-m \cdot n \in \mathbb{Z}_{0,+}$, hence $-(-m \cdot n)=m \cdot n \in \mathbb{Z}$ (since a number is in $\mathbb{Z}_{0,+}$ implies the negative of that number is in $\mathbb{Z}$).
Thus, $\mathbb{Z}$ is closed under multiplication.

Z4:The set of integers, $\mathbb{Z}$, is not closed under division.

We know that $2, 5 \in \mathbb{Z}$, but $\frac{2}{5}=0.4 \notin \mathbb{Z}$. Hence, $\mathbb{Z}$ is not closed under division.

2. We showed before (Section 3, Exercise#12) that
$\gcd(ma, mb) = m\gcd(a, b)$ if $m, a, b \in \mathbb{N}$

Theorem 5.7 states that for $a, b \in \mathbb{Z}$ with $a, b \neq 0$ we have

$$\gcd(a, b) = \gcd(-a, b) = \gcd(a, -b) = \gcd(-a, -b)$$

Let $m, a, b \in \mathbb{Z}$ with $m, a, b \neq 0$.
If $m, a, b \in \mathbb{N}$, then there is nothing to show.
Suppose $a, b < 0$.
$\gcd(ma, mb) = \gcd(-ma, -mb) = m\gcd(-a, -b) = m\gcd(a, b)$ (since $-ma > 0$ and $-mb > 0$).

Suppose $a < 0$ and $b > 0$.
$\gcd(ma, mb) = \gcd(-ma, mb) = m\gcd(-a, b) = m\gcd(a, b)$ (since $-ma \in \mathbb{N}$ and $mb \in \mathbb{N}$).

Hence, for $m, a, b \in \mathbb{Z}$ with $a, b \neq 0$ and $m > 0$, $\gcd(ma, mb) = m\gcd(a, b)$.

3. Let $a, b, c \in \mathbb{Z}$ with $a, b, c \neq 0$.
If $a, b, c \in \mathbb{N}$, then we know that $\gcd(\gcd(a, b), c) = \gcd(a, \gcd(b, c))$ (see Section 3, Exercise#13).
If not, then by using a similar argument to the previous exercise, we can get the result.
If $\gcd(a, b) < 0$ and $c < 0$, then since $-\gcd(a, b) > 0$ and $-c > 0$, by using the Exercise#13 and Theorem 5.7, we get;

$$\begin{aligned}\gcd(\gcd(a, b), c) &= \gcd(-\gcd(a, b), -c) \\ &= \gcd(-a, -\gcd(b, c)) = \gcd(a, \gcd(b, c))\end{aligned}$$

The other cases can be shown similarly.
Hence, $\gcd(\gcd(a, b), c) = \gcd(a, \gcd(b, c))$ for $a, b, c \in \mathbb{Z}$ with $a, b, c \neq 0$.

4. Let $a, b \in \mathbb{Z}$ with $a, b \neq 0$.
We showed before that for $a, b \in \mathbb{N}$, $\gcd(a, b) = \gcd(b, a)$ (see Section 3, Exercise#15).
Suppose $a, b < 0$. Then by Theorem 5.7 we have;
$\gcd(a, b) = \gcd(-a, -b) = \gcd(-b, -a) = \gcd(b, a)$ (since $-a, -b \in \mathbb{N}$)
Suppose $a < 0$ and $b > 0$. Then similarly,
$\gcd(a, b) = \gcd(-a, b) = \gcd(b, -a) = \gcd(b, a)$.

Hence, $\gcd(a,b) = \gcd(b,a)$.

5. We know that if $a,b \in \mathbb{N}$ then $\operatorname{lcm}(a, \gcd(a,b)) = a$ and $\gcd(a, \operatorname{lcm}(a,b)) = a$ (see Section 3, Exercise#16).

6. Follows from #5.

7. For $a,b \in \mathbb{N}$, we know that $\operatorname{lcm}(a,b) = \operatorname{lcm}(b,a)$.
Suppose $a,b < 0$. Then, again by theorem 5.7,
$\operatorname{lcm}(a,b) = \operatorname{lcm}(-a,-b) = lcm(-b,-a) = lcm(b,a)$ (since $-a,-b \in \mathbb{N}$).
Other cases follow similarly.
Hence, for $a,b \in \mathbb{Z}$, $\operatorname{lcm}(a,b) = \operatorname{lcm}(b,a)$.

8. Let $a,b,c \in \mathbb{Z}$ with $a,b,c \neq 0$. Show that
$\operatorname{lcm}(\operatorname{lcm}(a,b),c) = \operatorname{lcm}(a, \operatorname{lcm}(b,c))$

If $a,b,c \in \mathbb{N}$, then we showed before that
$\operatorname{lcm}(\operatorname{lcm}(a,b),c) = \operatorname{lcm}(a, \operatorname{lcm}(b,c))$ (see Section 3, Exercise#19).
Suppose $a,b,c < 0$. By Theorem 5.7, we get;

$$\begin{aligned}\operatorname{lcm}(\operatorname{lcm}(a,b),c) &= \operatorname{lcm}(\operatorname{lcm}(-a,-b),-c) = \operatorname{lcm}(-a, \operatorname{lcm}(-b,-c)) \\ &= \operatorname{lcm}(a, \operatorname{lcm}(b,c))\end{aligned}$$

The other cases follow with a similar argument.
Hence, for $a,b,c \in \mathbb{Z}$ with $a,b,c \neq 0$, $\operatorname{lcm}(\operatorname{lcm}(a,b),c) = \operatorname{lcm}(a, \operatorname{lcm}(b,c))$.

9. Let $m,a,b \in \mathbb{Z}$ with $a,b \neq 0,\ m > 0$. Show that $\operatorname{lcm}(ma,mb) = m\operatorname{lcm}(a,b)$
If $a,b \in \mathbb{N}$, then by Exercise #20 in Section 3, we know that
$\operatorname{lcm}(ma,mb) = m\operatorname{lcm}(a,b)$.
Suppose $a,b < 0$. Then
$\operatorname{lcm}(ma,mb) = \operatorname{lcm}(-ma,-mb) = m\operatorname{lcm}(-a,-b) = m\operatorname{lcm}(a,b)$.
You can show the other cases similarly.
Hence, for $m,a,b \in \mathbb{Z}$ with $a,b \neq 0,\ m > 0$, $\operatorname{lcm}(ma,mb) = m\operatorname{lcm}(a,b)$.

10. This problem is intended for extended exploration and discussion. There is no simple solution.

11.Let $\frac{a}{b}, \frac{c}{d}$ be any two positive rational numbers such that $\frac{a}{b} < \frac{c}{d}$. We know that $\frac{a}{b} = \frac{ad}{bd}$, and $\frac{c}{d} = \frac{cb}{db}$. It is important to observe that $ad < \frac{ad+cb}{2} < cb$.

Then the rational number $\frac{\frac{ad+cb}{2}}{bd}$ is a number between $\frac{ad}{bd}$ and $\frac{cb}{db}$. Hence,

$$\frac{a}{b} < \frac{\frac{ad+cb}{c}}{bd} < \frac{c}{d}.$$

We can conclude with a similar argument that same holds for negative rational numbers.

Hence, there is always a rational number between any two rational numbers.

12. Let $m, n, p, q, r, s \in \mathbb{Z},\ n, q, s \neq 0$, such that $a = \frac{m}{n}, b = \frac{p}{q}$ and $c = \frac{r}{s}$. Then $a, b, c \in \mathbb{Q}$.

i) Commutative Property:

$a + b = \frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq} = \frac{np+mq}{qn} = \frac{np}{qn} + \frac{mq}{qn} = \frac{p}{q} + \frac{m}{n} = b + a$ (notice that $np, mq, nq \in \mathbb{Z}$ and $\mathbb{Z}$ is commutative with respect to both addition and multiplication, i.e. $mq + np = np + mq$ and $np = pn$.)

$a \cdot b = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq} = \frac{pm}{qn} = \frac{p}{q} \cdot \frac{m}{n} = b \cdot a$ (since $\mathbb{Z}$ is commutative with respective to multiplication).

Hence, $\mathbb{Q}$ has commutative property both for addition and multiplication.

ii) Associative Property:

It is important to notice that $m, n, p, q, r, s \in \mathbb{Z}$, i.e. we can use the associative property of $\mathbb{Z}$ with respect to addition and multiplication and the distributive property.

$$(a+b)+c=\left(\frac{m}{n}+\frac{p}{q}\right)+\frac{r}{s}=\left(\frac{mq+np}{nq}\right)+\frac{r}{s}=\frac{mq+np}{nq}+\frac{r}{s}=\frac{(mq+np)s+r(nq)}{(nq)s}$$

$$=\frac{mqs+nps+rnq}{nqs}=\frac{mqs+(ps+rq)n}{n(qs)}=\frac{mqs}{n(qs)}+\frac{(ps+rq)n}{n(qs)}$$

$$=\frac{m}{n}+\left(\frac{ps+rq}{qs}\right)=\frac{m}{n}+\left(\frac{ps}{qs}+\frac{rq}{qs}\right)=\frac{m}{n}+\left(\frac{p}{q}+\frac{r}{s}\right)$$

$$=a+(b+c)$$

$$(a\cdot b)\cdot c=\left(\frac{m}{n}\cdot\frac{p}{q}\right)\cdot\frac{r}{s}=\left(\frac{mp}{nq}\right)\cdot\frac{r}{s}=\frac{mp}{nq}\cdot\frac{r}{s}=\frac{mpr}{nqs}=\frac{m(pr)}{n(qs)}$$

$$=\frac{m}{n}\cdot\left(\frac{pr}{qs}\right)=\frac{m}{n}\cdot\left(\frac{p}{q}\cdot\frac{r}{s}\right)$$

$$=a\cdot(b\cdot c)$$

Hence, $\mathbb{Q}$ has associative property both for addition and multiplication.

iii) Distributive Property:

$$a\cdot(b+c)=\frac{m}{n}\cdot\left(\frac{p}{q}+\frac{r}{s}\right)=\frac{m}{n}\cdot\left(\frac{ps+qr}{qs}\right)=\frac{m}{n}\cdot\frac{ps+qr}{qs}=\frac{m(ps+qr)}{n(qs)}$$

$$=\frac{mps+mqr}{nqs}=\frac{mps}{nqs}+\frac{mqr}{nqs}=\frac{mp}{nq}+\frac{mr}{ns}=\left(\frac{m}{n}\cdot\frac{p}{q}\right)+\left(\frac{m}{n}\cdot\frac{r}{s}\right)$$

$$=a\cdot b+a\cdot c$$

iv) Identity element:

Notice that for any non-zero integer n , $0=\frac{0}{n}$. Since for $m\in\mathbb{Z}$, $m+0=m$,

$a+0=\frac{m}{n}+\frac{0}{n}=\frac{m+0}{n}=\frac{m}{n}=a$.(By commutative property, $0+a=a+0=a$.)
Hence, 0 is the identity element of $\mathbb{Q}$ with respect to addition.
We also know that $1=\frac{1}{1}$ and for $m\in\mathbb{Z}$, $m\cdot 1=m$. $a\cdot 1=\frac{m}{n}\cdot\frac{1}{1}=\frac{m\cdot 1}{n\cdot 1}=\frac{m}{n}=a$
(By the commutative property with respect to multiplication, $1.a=a.1=a$.)
Hence, 1 is the identity element of $\mathbb{Q}$ with respect to multiplication.

v) Inverse element:

We know that for $m \in \mathbb{Z}$, there exists $-m \in \mathbb{Z}$ such that $m+(-m)=0$. Let $\frac{-m}{n}=-a; \quad -a \in \mathbb{Q}$. Then,

$$a+(-a)=\frac{m}{n}+\frac{-m}{n}=\frac{m+(-m)}{n}=\frac{0}{n}=0.$$

Hence, for every $a \in \mathbb{Q}$, there exists $-a \in \mathbb{Q}$ such that $a+(-a)=(-a)+a=0$.

Let a be a non-zero element of $\mathbb{Q}$, i.e. $a=\frac{m}{n}$, where $m,n \in \mathbb{Z}, m \neq 0, n \neq 0$.

Notice that $\frac{m}{n} \cdot \frac{n}{m}=\frac{mn}{nm}=1$. Let $a^{-1}=\frac{n}{m}; \quad a^{-1} \in \mathbb{Q}$. Then $a \cdot a^{-1}=a^{-1} \cdot a=1$.

Hence, for every $a \in \mathbb{Q}$, there exists $a^{-1} \in \mathbb{Q}$ such that $a \cdot a^{-1}=a^{-1} \cdot a=1$.

That is, every rational number has additive inverse, and every non-zero rational number has multiplicative inverse.

13. $\frac{2}{29}=0.\overline{0689655172413793103448275862 0}$ (The division is left as an exercise).