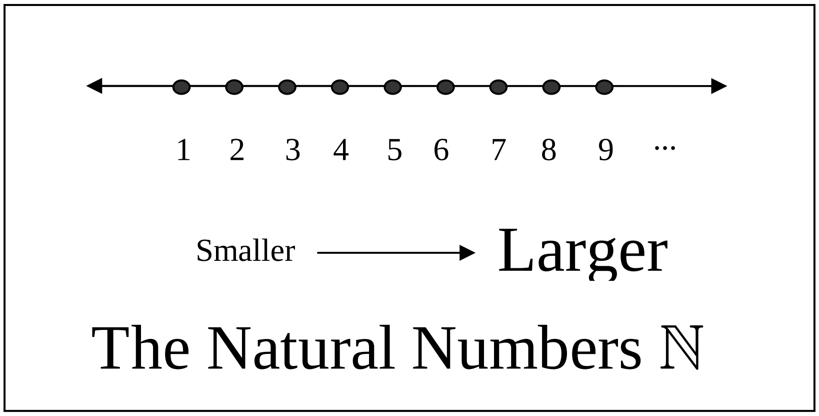
1. The Natural Numbers

Definition 1.1: The set of **natural numbers**, $\mathbb{N}$, is given by

$$\mathbb{N} = \{\ 1,\ 2,\ 3,\ 4,\ \ldots\ \}$$

$\mathbb{N}$ is a subset of the real numbers, and we visualize $\mathbb{N}$ as a set of infinitely many isolated points which are equally spaced along the real number line.



A list of basic facts associated with $\mathbb{N}$ are given below:

N1. $\mathbb{N}$ is *closed under addition*; i.e. the sum of two natural numbers is always a natural number.
N2. $\mathbb{N}$ is the smallest subset of real numbers which contains 1 and is closed under addition.
N3. $\mathbb{N}$ is *closed under multiplication*; i.e. the product of any two natural numbers is a natural number.
N4. $\mathbb{N}$ is not closed under subtraction or division since $13-23$ is not a natural number and $3 \div 7$ is not a natural number.
N5. $\mathbb{N}$ does not contain the *additive identity* 0.
N6. $\mathbb{N}$ contains the *multiplicative identity* 1.
N7. $\mathbb{N}$ contains a smallest element; namely 1.
N8. If $r \in \mathbb{R}$ then there is an element $n \in \mathbb{N}$ such that $r < n$.

Definition 1.2: Let $m, n \in \mathbb{N}$. We say that ***m* is a multiple of *n*** if and only if there is a natural number *k* so that $m = kn$.

Definition 1.3: We say that a subset *S* of $\mathbb{R}$ is closed under addition if and only if $a+b \in S$ whenever $a, b \in S$. *S* is closed under multiplication if and only if $ab \in S$ whenever $a, b \in S$. *S* is closed under division if and only if $a/b \in S$ whenever $a, b \in S$ and $b \neq 0$. *S* is closed under subtraction if and only if $a-b \in S$ whenever $a, b \in S$.

Example 1.4: Let $S = \{ 3n \mid n \in \mathbb{N} \}$, the set of positive multiples of 3. Notice that

$$S = \{ 3, 6, 9, 12, ... \}$$

Show that *S* is closed under both addition and multiplication.

Solution: We start by picking two numbers $3m$ and $3n$ in *S*. Then

$$3m + 3n = 3(m+n) \text{ from the distributive property.}$$

Therefore, $3m + 3n$ is a multiple of 3, since $m + n \in \mathbb{N}$ from **N1**. Also,

$$(3m)(3n) = 3(3mn) \text{ from the associative and commutative properties.}$$

Therefore, $(3m)(3n)$ is a multiple of 3, since $3mn \in \mathbb{N}$ from **N3**. So, *S* is closed under both addition and multiplication.

We assume the ordering of the natural numbers is well understood. We know that

$$1 < 2 < 3 < 4 < ...$$

Of course, there are many natural numbers which are much more complicated than the ones shown above, and it is not always a trivial matter to see (on first inspection) how two natural numbers are ordered.

Example 1.5: Which of the following is true,

$$341789312633 > 67325698994 \text{ or } 341789312633 < 67325698994 \text{ ?}$$

Solution: If you check carefully, you will see that the natural number 341789312633 has more digits than the natural number 67325698994. Consequently,

$$341789312633 > 67325698994$$

If two natural numbers have the same number of digits, then the digits can be used to determine which number is larger.

> **N9.** A natural number *x* is larger than a natural number *y* if *x* has more digits than *y*. If *x* and *y* have the same number of digits, then the size is determined by comparing the digits from left to right.

Example 1.6: Which of the natural numbers 345278936 and 345728936 is larger?

Solution: Each of these natural numbers have 9 digits. Comparing the first four digits from left to right we see 3=3, 4=4, 5=5 and 2<7. Consequently,

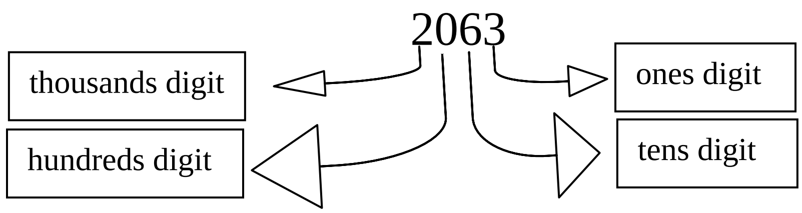
345278936 < 345728936.

The digits in a natural number have special meaning. For example,

$$354 = 3 \times 100 + 5 \times 10 + 4 \times 1$$

$$2063 = 2 \times 1000 + 0 \times 100 + 6 \times 10 + 3 \times 1$$

This gives rise to the terms *ones-digit* or *ones-place, tens-digit* or *tens-place, hundreds-digit* or *hundreds-place, thousands-digit* or *thousands-place,* etc.



There are a number of important subsets of $\mathbb{N}$.

Definition 1.7: The **even natural numbers** are the natural numbers which are multiples of 2. The **odd natural numbers** are the natural numbers which are **not** multiples of 2.

We can see that the set of even natural numbers is given by $\{2, 4, 6, 8, 10, 12, ...\}$ and that the set of even natural numbers can also be described as $\{\, 2n \mid n \in \mathbb{N}\}$. You should be able to show that a natural number is even if its ones digit is either 0, 2, 4, 6 or 8.

Similarly, we can see that the set of odd natural numbers is given by $\{1, 3, 5, 7, 9, 11, ...\}$ and the set of odd natural numbers can also be described as $\{\, 2n+1 \mid n \in \mathbb{N} \text{ or } n = 0\}$. Notice that from Definition 5, every natural number is either an even number or an odd number.

Example 1.8: The even natural numbers are closed under multiplication.

Solution: We start by choosing two arbitrary even natural numbers. We can write these as $2m$ and $2n$ where $m, n \in \mathbb{N}$. Then

$$\begin{aligned}(2m)(2n) &= 4mn \\ &= 2(2mn)\end{aligned}$$

by the associative and commutative properties. From **N3**, $2mn$ is a natural number. So, the product $(2m)(2n)$ is an even number. Therefore, the even natural numbers are closed under multiplication.

Example 1.9: The even natural numbers are closed under addition.

Solution: We start by choosing two arbitrary even natural numbers. We can write these as $2m$ and $2n$ where *m* and *n* are natural numbers. Then

$$2m + 2n = 2(m+n)$$

from the distributive property. **N1** implies $m+n$ is an integer. So, the sum $2m+2n$ is an even natural number. Therefore, the even natural numbers are closed under addition.

Exercises

1. Which number is larger, 3278456091 or 452901362?
2. Which number is larger, 42567893456 or 42567983456?
3. Determine whether the odd natural numbers are closed under addition.
4. Determine whether the even natural numbers are closed under subtraction.
5. Let $S = \left\{2^n \,\middle|\, n = 0 \text{ or } n \in \mathbb{N}\right\}$. Show that S is closed under multiplication. Determine whether this set is closed under addition.
6. Show that the odd natural numbers are closed under multiplication.
7. Use properties **N8** and **O1-O4** to show that if $a \in \mathbb{R}$ with $a > 0$ then there is an element $m \in \mathbb{N}$ so that $0 < \frac{1}{m} < a$.

We gave a simple criteria above for checking whether a natural number is a multiple of 2. Namely, the ones digit is one of 0, 2, 4, 6 or 8. It is also possible to give criteria to check whether a natural number is a multiple of 3, 4, 5, 6, 7, 8, 9 or 10. We state the result for multiples of 3 as a theorem, and place the other results in the exercises.

Theorem 1.10: A number $n \in \mathbb{N}$ is a multiple of 3 if and only if the sum of its digits is a multiple of 3.

Proof: Suppose the number n has the form

$$n = a_k a_{k-1} \ldots a_1 a_0$$

where the numbers a_k, a_{k-1}, ... a_1, a_0 are the digits of n. Then

$$\begin{aligned} n &= a_k\left(10^k\right) + a_{k-1}\left(10^{k-1}\right) + \cdots + a_1(10) + a_0 \\ &= a_k\left(\underbrace{99\ldots9}_{k \text{ times}} + 1\right) + a_{k-1}\left(\underbrace{99\ldots9}_{k-1 \text{ times}} + 1\right) + \cdots + a_1(9+1) + a_0 \\ &= a_k\left(\underbrace{99\ldots9}_{k \text{ times}}\right) + a_{k-1}\left(\underbrace{99\ldots9}_{k-1 \text{ times}}\right) + \cdots + a_1(9) + \left(a_k + a_{k-1} + \cdots + a_1 + a_0\right) \end{aligned}$$

Since $a_k\left(\underbrace{99\ldots9}_{k \text{ times}}\right) + a_{k-1}\left(\underbrace{99\ldots9}_{k-1 \text{ times}}\right) + \cdots + a_1(9)$ is a multiple of 3, we find that n is a multiple of 3 if and only if $\left(a_k + a_{k-1} + \cdots + a_1 + a_0\right)$ is a multiple of 3.

Example 1.11: The proof technique above might not seem transparent unless we demonstrate it with a specific example. Notice that

$$\begin{aligned} 24654 &= 2(10000) + 4(1000) + 6(100) + 5(10) + 4 \\ &= 2(9999+1) + 4(999+1) + 6(99+1) + 5(9+1) + 4 \\ &= 2(9999) + 4(999) + 6(99) + 5(9) + (2+4+6+5+4) \end{aligned}$$

The number $2(9999)+4(999)+6(99)+5(9)$ is a multiple of 3. So, 24654 is a multiple of 3 if and only if 2 + 4 + 6 + 5 + 4 is a multiple of 3. This sum is 21, which is a multiple of 3. Therefore, 24654 is a multiple of 3.

Example 1.12: The number 183 is a multiple of 3, since the sum of its digits is

$$1 + 8 + 3 = 12,$$

which is a multiple of 3. The number 9763218 is a multiple of 3, since the sum of its digits is

$$9 + 7 + 6 + 3 + 2 + 1 + 8 = 36,$$

which is a multiple of 3. The number 5167283 is not a multiple of 3 since the sum of its digits is 32, which is not a multiple of 3.

Exercises

1. Show that the number 111234546327 is a multiple of 3.
2. Show that the number 234567890 is not a multiple of 3.
3. Let $a, b, c \in \mathbb{N}$ with $a = b + c$, and suppose *b* is a multiple of 3. Show *a* is a multiple of 3 if and only if *c* is a multiple of 3.
4. This is a generalization of the exercise above. Let $a, b, c, d \in \mathbb{N}$ with $a = b + c$, and suppose *b* a multiple of *d*. Show *a* is a multiple of *d* if and only if *c* is a multiple of *d*.
5. Show that a natural number is a multiple of 9 if and only if its digits sum to a multiple of 9.
6. Show that the number 4545454563 is a multiple of 9.
7. A natural number is a multiple of 4 if and only if the number formed by the last two digits is a multiple of 4. (Hint: 100, 1000, 10000, etc. are all multiples of 4.)
8. Which of the numbers 234516, 324414 and 2314856 are multiples of 4?
9. Show that a natural number is a multiple of 5 if and only if its ones digit is 0 or 5.
10. Show that a natural number is a multiple of 6 if and only if it is ones digit is 0, 2, 4, 6 or 8, and the sum of its digits is a multiple of 3.
11. Explain why the exercise above is equivalent to saying that a natural number is a multiple of 6 if and only if it is a multiple of both 2 and 3.
12. Show that a natural number is a multiple of 8 if and only if the number formed by the last three digits is a multiple of 8. (Hint: 1000, 10000, 100000, etc. are multiples of 8.)
13. Which of the numbers 234516, 324414 and 2314856 are multiples of 8?
14. Show that a natural number is a multiple of 10 if and only if the ones digit is 0.
15. Show that a natural number, written as $a_n a_{n-1} \cdots a_2 a_1 a_0$ is a multiple of 7 if and only if $\underbrace{22\cdots2}_{n-1 \text{ times}}3a_n + \underbrace{22\cdots2}_{n-2 \text{ times}}3a_{n-1} + \cdots + 23a_2 + 3a_1 + a_0$ is a multiple of 7.
16. Let $m, n \in \mathbb{N}$. We say that ***m* is divisible by *n*** if and only if *m* is a multiple of *n*. We say that ***n* divides *m*** if and only if *m* is divisible by *n*. Restate the exercises above using this new language.