Square Off!

Purpose:
Participants will determine the GCD for a set of numbers using geometric models.

Overview:

Participants will determine which carpet squares will completely carpet a rectangular room with given dimensions. They will explain the connection between the size of the carpet square and the GCD for the dimensions of the room. Then participants will be given a rectangular model of a room subdivided into squares and asked to explain how this model will help determine the GCD for the two numbers that represent the numerical part of the dimensions using a method called "square off".

TExES Mathematics 4-8 Competencies. The beginning teacher:

I.001.C Demonstrates an understanding of a variety of models for representing numbers (e.g., fraction strips, diagrams, patterns, shaded regions, number lines).

I.001.F Understands the characteristics of the set of whole numbers, integers, rational numbers, real numbers, and complex numbers (e.g., commutativity, order, closure, identity elements, inverse elements, density).

I.003.A Demonstrates an understanding of ideas from number theory (e.g., prime factorization, greatest common divisor) as they apply to whole numbers, integers, and rational numbers, and uses these ideas in problem situations.

I.003.B Uses integers, rational numbers, and real numbers to describe and quantify phenomena such as money, length, area, volume, and density.

I.003.E Applies properties of the real numbers to solve a variety of theoretical and applied problems.

V.016.A Recognizes and uses multiple representations of a mathematical concept (e.g., a point and its coordinates, the area of circle as a quadratic function in r, probability as the ratio of two areas).

TEKS Mathematics Objectives. The student is expected to:

4.3.A, 5.3.A Use addition and subtraction to solve problems involving whole numbers.

4.4.A Model factors and products using arrays and area models.

4.4.B Represent multiplication and division situations in picture, word, and number form.

4.4.D Use multiplication to solve problems involving two-digit numbers.

5.3.B Use multiplication to solve problems involving whole numbers (no more than three digits times two digits without technology).

5.3.D Identify prime factors of a whole number and common factors of a set of whole numbers.

6.1.E Identify factors and multiples including common factors and common multiples.

6.2.C Use multiplication and division of whole numbers to solve problems.

7.1.C Represent squares and square roots using geometric models.

7.2.E Simplify numerical expressions involving order of operations and exponents.

7.2.F, 8.2.A Select and use appropriate operations to solve problems and justify the selections.

Terms.
Rectangle, area, product, square, factor, common factor, common divisor, greatest common divisor, area model, rectangular array

Materials.
- Transparencies
- Activity Sheets
- Calculators
- Scissors
- Centimeter grid paper

Transparencies.
- *Transparency 1: Square Off!*
- *Transparency 2: Square Off!*

Activity Sheet(s).
- *Activity Sheet 1: Square Off!*
- *Activity Sheet 2: Square Off!*

Procedure:

Steps	Questions/Math Notes
1. Place the Transparency *Square Off!* on the overhead projector and have participants read the problem.	*When you are asked to "square off" a rectangular region, what does that mean?* *If a carpet tile is a 3 ft. x 3 ft. square and tiles of this size can completely cover a region, then what is the relationship between the 3 ft. x 3 ft. square and the area of that region?*
2. Have participants work with a partner on Activity Sheet 1 *Square Off!* using centimeter grid paper to model the problem. Monitor their work and ask scaffolding questions as needed to clarify and extend their thinking about the GCD and how to model it.	*How did you determine which number(s) met the conditions of the problem using your model?* *Are there any possible dimensions that you did not* *use? Explain your reasoning.* *How did you organize your work to determine the possible dimensions of the playroom?*
3. Debrief Activity 1 *Square Off!* by having several groups share how they solved the problems. It is important that participants make the connection between "square off" or covering a rectangular region with squares of a given dimension and the GCD for the numerical values of those dimensions.	*How did you determine possible dimensions of the* *playroom?* *How do these dimensions meet the conditions of the problem?*

4. Display Transparency 2 *Square Off!* for participants to read.	
5. Have participants work independently for about 5 minutes on the Activity Sheet 2 problem. Then have them put "heads together" to discuss possible solution(s). Emphasize to participants that this strategy is a good one to use with students working in groups. It increases the accountability for the work and provides an opportunity for each member of the group to make a contribution of ideas for solving the problem. The focus of this activity should be on the mathematics involved in the procedure or method of "squaring off".	*How did "putting heads together" help facilitate the group discussion?* *What have you learned by listening to your colleagues as they shared ideas on solving the problem?* *How are you using the model given?*
6. After participants have solved the problem, ask some of the groups with different solution strategies to write their solution(s) on an over-head transparency. Lead a whole group discussion about the problem and the mathematics used to solve the problems on Activity Sheet 2.	*How did the model help you find the GCD of the numerical part of the dimensions of the rectangle?* *How did your group explain the mathematics Involved in this process of "squaring off"?*

Possible Solution:

Activity Sheet 1

1. To determine either dimension of the playroom, find dimensions with factors of 1, 2, and 4 and no factors of 3 or 5. Since 2 is a factor of 4, we can find multiples of 4 that do not have factors of 3 or 5. The following shows multiples of 4 that satisfy the conditions:

4x1 = 4
4x2 = 8
4x4 =16
4x7 =28
4x8= 32

The products above represent some of the numbers that meet the conditions of the problem. Some possible dimensions of the room could be as follows:

4 ft x 8 ft	8 ft x 16 ft	16 ft x 16 ft
4 ft x 16 ft	8 ft x 28 ft	16 ft x 28 ft
4 ft x 28 ft	8 ft x 32 ft	16 ft x 32 ft

2. Answers will vary. The idea here is for participants to use squares with given dimensions to "*square off*" a rectangular region. If a 3 ft square will "*square off*" a rectangular region, then 3 is a factor of each of the dimensions of the rectangle. If a 6 ft square is the largest square that will "square off" a rectangular region, then 6 is called the GCD of the two numbers that make up the numerical part of its dimensions. This part of the activity will enable participants to make a connection between a geometric model and the GCD of the two numbers in the dimensions.

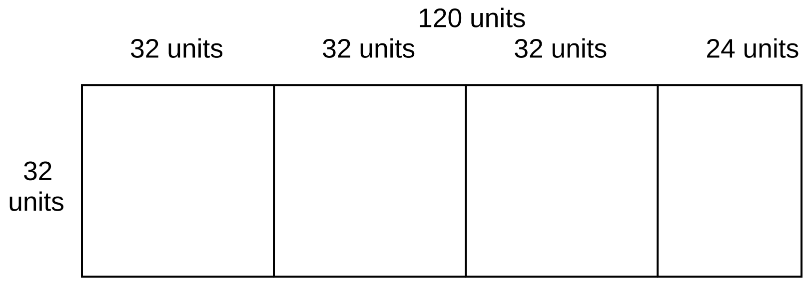
3. Answers will vary. Example: A playroom with dimensions of 16 ft by 28 ft
4 ft square tiles would be the largest size for a square tile that would completely cover a floor with these dimensions and would represent the GCD of the numerical part of its dimensions. 1 ft and 2 ft square tiles would also completely cover the floor, but these would not represent the largest size.

Activity Sheet 2

1. The area of the large rectangle equals the sum of the areas of the squares into which it has been subdivided. The length of 120 un can be expressed as (32 + 32 + 32 + 24) and the area of the large rectangle is as follows:

$$\begin{aligned} A &= 32\,(120) \\ &= \mathbf{32}\,(32 + 32 + 32 + 24) \\ &= \mathbf{32}\bullet 32 + \mathbf{32}\bullet 32 + \mathbf{32}\bullet 32 + \mathbf{32}\bullet 24 \quad \text{(distributive property)} \end{aligned}$$

↓ ↓ ↓ ↓
cut off cut off cut off use to form more squares

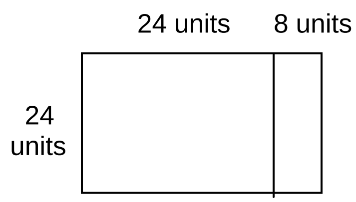


GCD (32, 120) = GCD (24, 32)

After cutting off the three squares (32 units x 32 units), there is one rectangle left with dimensions 32 units by 24 units. This rectangle still has a side length of 32 units equal to a side length of the squares removed. 32 is a common factor of each of the partial products used to obtain the area of the entire rectangular region. So it is possible to work with only the remaining rectangle with dimensions 32 units by 24 units.

$$\begin{aligned} B &= 24\,(32) \\ &= \mathbf{24}\,(24 + 8) \\ &= \mathbf{24}\bullet 24 + \mathbf{24}\bullet 8 \quad \text{(distributive property)} \end{aligned}$$

↓ ↓
cut off use to form more squares

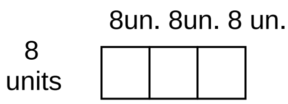


GCD (24, 32) = GCD (8, 24)

Now, divide the remaining rectangle into square(s) with side length 24 units. There is only one with a small rectangle at the bottom having dimensions 8 units by 24 units (diagram on Activity Sheet 2). 24 is not a factor of 32, since it is not possible to "square off" the rectangle with a 24 unit square. Cut off the square with 24 units on each side. Divide the remaining rectangle (24 units by 8 units) into squares 8 units by 8 units. Notice this remaining rectangle has dimensions that are factors of both 32 and 24. This means that an 8 unit square can be used to "square off" the rectangular region with dimensions 32 units by 120 units.

$$\begin{aligned} C &= 8\,(24) \\ &= \mathbf{8}\,(8 \quad + \quad 8 \quad + \quad 8) \\ &= \mathbf{8}\bullet 8 \quad + \quad \mathbf{8}\bullet 8 \quad + \quad \mathbf{8}\bullet 8 \qquad \text{(distributive property)} \end{aligned}$$

↓ cut off ↓ cut off ↓ cut off



Notice the tie to the Euclidean Algorithm. Using the Euclidean Algorithm to find the GCD (120, 32), we get

$R_1 = 120 - (3)(32) = 24$ or $120 = 3(32) + 24$

$R_2 = 32 - (1)(24) = 8$ $\quad$ $32 = 1(24) + 8$

$R_3 = 24 - (3)(8) = 0$ $\quad$ $24 = 3(8) + 0$

Consequently, GCD (120, 32) = 8

2. Answers will vary.

54 = 3(15) + 9	or	GCD (15, 54) =
15 = 1(9) + 6		GCD (9, 15) =
9 = 1(6) + 3		GCD (6, 9) =
6 = 2(3) + 0		GCD (3, 6) = 3
GCD (15, 54) = 3		

References.

Bennett, A., & Foreman, L. (1991). Student Activity 44B. *Visual Mathematics*. Salem, OR: The Math Learning Center.