2.4 Some Applications

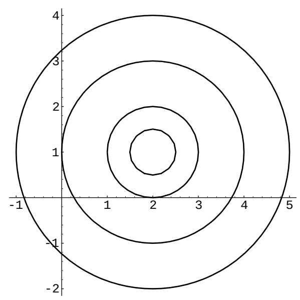
In this section we give some examples of applications of first order differential equations.

1. Orthogonal Trajectories

The one-parameter family of curves

$$(x-2)^2 + (y-1)^2 = C \qquad (C \geq 0) \tag{a}$$

is a family of circles with center at the point $(2,1)$ and radius $\sqrt{C}$.



If we differentiate this equation with respect to x, we get

$$2(x-2) + 2(y-1)\, y' = 0$$

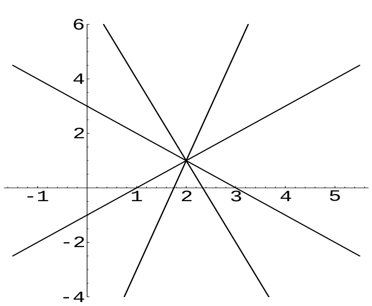
and

$$y' = -\frac{x-2}{y-1} \tag{b}$$

This is the differential equation for the family of circles. Note that if we choose a specific point (x_0, y_0), $y_0 \neq 1$ on one of the circles, then (b) gives the slope of the tangent line at (x_0, y_0).

Now consider the family of straight lines passing through the point $(2,1)$:

$$y - 1 = K(x-2). \tag{c}$$

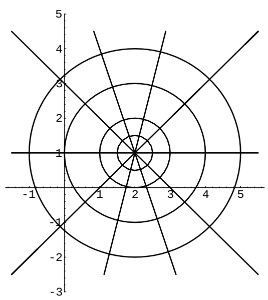


The differential equation for this family is

$$y' = \frac{y-1}{x-2} \qquad \text{(verify this)} \tag{d}$$

Comparing equations (b) and (d) we see that right side of (b) is the negative reciprocal of the right side of (d). Therefore, we can conclude that if $P\,(x_0, y_0)$ is a point of intersection of one of the circles and one of the lines, then the line and the circle are perpendicular (orthogonal) to each other at P. Of course we already knew this from plane geometry: a radius and a tangent are perpendicular to each other at the point of tangency.

The following figure shows the two families drawn in the same coordinate system.



A curve that intersects each member of a given family of curves at right angles (orthogonally) is called an *orthogonal trajectory* of the family. Each line in (c) is an orthogonal trajectory of the family of circles (a) [and conversely, each circle in (a) is an orthogonal trajectory of the family of lines (c)]. In general, if

$$F(x, y, c) = 0 \qquad \text{and} \qquad G(x, y, K) = 0$$

are one-parameter families of curves such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be *orthogonal trajectories.*

A procedure for finding a family of orthogonal trajectories $G(x, y, K) = 0$ for a given family of curves $F(x, y, C) = 0$ is as follows:

Step 1. Determine the differential equation for the given family $F(x, y, C) = 0$.

Step 2. Replace y' in that equation by $-1/y'$; the resulting equation is the differential equation for the family of orthogonal trajectories.

Step 3. Find the general solution of the new differential equation. This is the family of orthogonal trajectories.

Example Find the orthogonal trajectories of the family of parabolas $y = Cx^2$.

SOLUTION You can verify that the differential equation for the family $y = Cx^2$ can be written as

$$y' = \frac{2y}{x}.$$

Replacing y' by $-1/y'$, we get the equation

$$-\frac{1}{y'} = \frac{2y}{x} \qquad \text{which simplifies to} \qquad y' = -\frac{x}{2y}$$

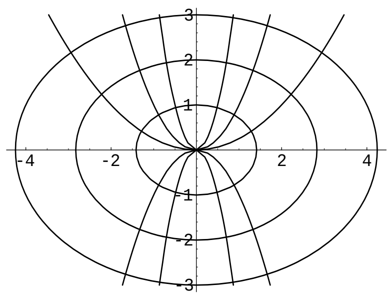
a separable equation. Separating the variables, we get

$$2y\, y' = -x \qquad \text{or} \qquad 2y\, dy = -x\, dx.$$

Integrating with respect to x, we have

$$y^2 = -\frac{1}{2}\, x^2 + C \qquad \text{or} \qquad \frac{x^2}{2} + y^2 = C.$$

This is a family of ellipses with center at the origin and major axis on the x-axis. ■



Exercises 2.4.1

Find the orthogonal trajectories for the family of curves.

1. $y = Cx^3$.
2. $x = Cy^4$.
3. $y = Cx^2 + 2$.
4. $y^2 = 2(C - x)$.
5. $y = C \cos x$
6. $y = Ce^x$
7. $y = \ln(Cx)$
8. $(x + y)^2 = Cx^2$

Find the orthogonal trajectories for the family of curves.

9. The family of parabolas symmetric with respect to the x-axis and vertex at the origin.

10. The family of parabolas with vertical axis and vertex at the point $(1, 2)$.

11. The family of circles that pass through the origin and have their center on the x-axis.

12. The family of circles tangent to the x-axis at $(3, 0)$.

Show that the given family is *self-orthogonal.*

13. $y^2 = 4C(x + C)$.

14. $\dfrac{x^2}{C^2} + \dfrac{y^2}{C^2 - 4} = 1$.

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2. Exponential Growth and Decay

Radioactive Decay: It has been observed and verified experimentally that the rate of decay of a radioactive material at time t is proportional to the amount of material present at time t. Mathematically this says that if $A = A(t)$ is the amount of radioactive material present at time t, then

$$\frac{dA}{dt} = rA$$

where r, the constant of proportionality, is negative. To emphasize the fact that A is decreasing, this equation is often written

$$\frac{dA}{dt} = -kA \qquad \text{or} \qquad \frac{dA}{dt} = -kA, \quad k > 0 \text{ constant.}$$

This is the form we shall use. The constant of proportionality k is called the *decay constant*
.

Note that this equation is both linear and separable and so we can use either method to solve it. It is easy to show that the general solution is

$$A(t) = Ce^{-kt}$$

If $A_0 = A(0)$ is amount of material present at time $t = 0$, then $C = A_0$ and

$$A(t) = A_0\, e^{-kt}.$$

Note that $\lim_{t \to \infty} A(t) = 0$.

Half-Life: An important property of a radioactive material is the length of time T it takes to decay to one-half the initial amount. This is the so-called *half-life* of the material. Physicists and chemists characterize radioactive materials by their half-lives. To find T we solve the equation

$$\tfrac{1}{2}\, A_0 = A_0\, e^{-kT}$$

for T:

$$\begin{aligned}
\tfrac{1}{2}\,A_0 &= A_0\,e^{-kT}\\
e^{-kT} &= \tfrac{1}{2}\\
-kT &= \ln(1/2) = -\ln 2\\
T &= \frac{\ln 2}{k}
\end{aligned}$$

Conversely, if we know the half-life T of a radioactive material, then the decay constant k is given by

$$k = \frac{\ln 2}{T}.$$

Example Cobalt-60 is a radioactive element that is used in medical radiology. It has a half-life of 5.3 years. Suppose that an initial sample of cobalt-60 has a mass of 100 grams.

(a) Find the decay constant and determine an expression for the amount of the sample that will remain t years from now.

(b) How long will it take for 90% of the sample to decay?

SOLUTION (a) Since the half-life $T = (\ln 2)/k$, we have

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5.3} \cong 0.131.$$

With $A(0) = 100$, the amount of material that will remain after t years is

$$A(t) = 100\,e^{-0.131t}.$$

(b) If 90% of the material decays, then 10%, which is 10 grams, remains. Therefore, we solve the equation

$$100\,e^{-0.131t} = 10$$

for t:

$$e^{-0.131t} = 0.1, \qquad -0.131t = \ln(0.1), \qquad t = \frac{\ln(0.1)}{-0.131} \cong 17.6.$$

It will take approximately 17.6 years for 90% of the sample to decay. ■

Population Growth; Growth of an Investment: It has been observed and verified experimentally that, under ideal conditions, a population (e.g., bacteria, fruit flies, humans, etc.) tends to increase at a rate proportional to the size of the population. Therefore, if $P = P(t)$ is the size of a population at time t, then

$$\frac{dP}{dt} = rP, \quad r > 0 \text{ (constant)} \tag{1}$$

In this case, the constant of proportionality r is called the *growth constant* .

Similarly, in a bank that compounds interest continuously, the rate of increase of funds at time t is proportional to the amount of funds in the account at time t. Thus equation (1) also represents the growth of a principal amount under continuous compounding. Since the two cases are identical, we'll focus on the population growth case.

The general solution of equation (1) is

$$P(t) = Ce^{rt}.$$

If $P(0) = P_0$ is the size of the population at time $t = 0$, then

$$P(t) = P_0\, e^{rt}$$

is the size of the population at time t. Note that $\lim_{t\to\infty} P(t) = \infty$. In reality, the rate of increase of a population does not continue to be proportional to the size of the population. After some time has passed, factors such as limitations on space or food supply, introduction of diseases, and so forth, affect the growth rate; the mathematical model is not valid indefinitely. In contrast, the model does hold indefinitely in the case of the growth of an investment under continuous compounding.

Doubling time: The analog of the half-life of a radioactive material is the so-called *doubling time,* the length of time T that it takes for a population to double in size. Using the same analysis as above, we have

$$\begin{aligned} 2\,A_0 &= A_0\, e^{rT} \\ e^{rT} &= 2 \\ rT &= \ln 2 \\ T &= \frac{\ln 2}{r} \end{aligned}$$

In the banking, investment, and real estate communities there is a standard measure, called the *rule of 72*, which states that the length of time (approximately) for a principal invested at $r\%$, compounded continuously, to double in value is $72/r\%$.

We know that the doubling time is

$$T = \frac{\ln 2}{r} \approx \frac{0.69}{r} = \frac{69}{r\%} \approx \frac{72}{r\%}.$$

This is the origin of the "rule of 72;" 72 is used rather than 69 because it has more divisors. ■

Example Scientists have observed that a small colony of penguins on a remote Antarctic island obeys the population growth law. There were 2000 penguins initially and 3000 penguins 4 years later.

(a) How many penguins will there be after 10 years?

(b) How long will it take for the number of penguins to double?

SOLUTION Let $P(t)$ denote the number of penguins at time t. Since $P(0) = 2000$ we have

$$P(t) = 2000\, e^{rt}.$$

We use the fact that $P(4) = 3000$ to determine the growth constant r:

$$3000 = 2000\, e^{4r}, \qquad e^{4r} = 1.5, \qquad 4r = \ln 1.5,$$

and so

$$r = \frac{\ln 1.5}{4} \cong 0.101.$$

Therefore, the number of penguins in the colony at any time t is

$$P(t) = 2000\, e^{0.101t}.$$

(a) The number of penguins in the colony after 10 years is (approximately)

$$P(10) = 2000\, e^{(0.101)10} = 2000\, e^{1.01} \cong 5491.$$

(b) To find out how long it will take the number of penguins in the colony to double, we need to solve

$$2000\, e^{0.101t} = 4000$$

for t:

$$e^{0.101t} = 2, \qquad 0.101t = \ln 2, \qquad t = \frac{\ln 2}{0.101} \cong 6.86 \text{ years.}$$

NOTE: There is another way of expressing P that uses the exact value of r. From the equation $3000 = 2000\, e^{4r}$ we get $r = \frac{1}{4} \ln \frac{3}{2}$. Thus

$$P(t) = 2000\, e^{\frac{t}{4} \ln [3/2]} = 2000\, e^{\ln [3/2]^{t/4}} = 2000 \left(\frac{3}{2}\right)^{t/4}. \qquad ■$$

Exercises 2.4.2

1. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:
   (a) An expression for the mass of the material remaining at any time t.
   (b) The mass of the material after 4 hours.
   (c) The half-life of the material.

2. What is the half-life of a radioactive substance if it takes 5 years for one-third of the material to decay?

3. The half-life of a certain radioactive material is 2 hours. How long will it take for a given amount of the material to decay to 1/10 of its original mass?

4. The half-life of radium-226 is 1620 years.
   (a) If an original sample of 100 grams of radium-226 was present initially, how much will remain after 500 years?
   (b) How long will it take for the sample to be reduced to 25 grams?

5. The size of a certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of 0.25 square centimeters initially, and after 8 hours it occupied an area of 0.35 square centimeters.
   (a) Estimate the size of the colony t hours after the initial measurement.
   (b) What is the expected size of the colony after 12 hours?
   (c) Find the doubling time of the colony.

6. A biologist observes that a certain bacterial colony triples every 4 hours and after 12 hours occupies 1 square centimeter.
   (a) How much area did the colony occupy when first observed?
   (b) What is the doubling time for the colony?

7. In 1980 the world population was approximately 4.5 billion and in the year 2000 it was approximately 6 billion. Assume that the world population at each time t increases at a rate proportional to the population at time t. Measure t in years after 1980.
   (a) Find the growth constant and give the world population at any time t.
   (b) How long will it take for the world population to reach 9 billion (double the 1980 population)?
   (c) The world population for 2002 was reported to be about 6.2 billion. What population does the formula in (a) predict for the year 2002?

8. It is estimated that the arable land on earth can support a maximum of 30 billion people. Extrapolate from the data given in Exercise 7 to estimate the year when the food supply becomes insufficient to support the world population.

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3. Newton's Law of Cooling/Heating

Newton's Law of Cooling states that the rate of change of the temperature u of an object is proportional to the difference between u and the (constant) temperature σ of the surrounding medium (e.g., air or water), called the *ambient temperature.* The mathematical formulation (model) of this statement is:

$$\frac{du}{dt} = m(u - \sigma), \quad m \text{ constant.}$$

The constant of proportionality, m, in this model must be negative; for if the object is warmer than the ambient temperature $(u - \sigma > 0)$, then its temperature will decrease $(du/dt < 0)$, which implies $m < 0$; if the object is cooler than the ambient temperature $(u - \sigma < 0)$, then its temperature will increase $(du/dt > 0)$, which again implies $m < 0$.

To emphasize that the constant of proportionality is negative, we write Newton's Law of Cooling as

$$\frac{du}{dt} = -k(u - \sigma), \quad k > 0 \text{ constant.} \tag{1}$$

This differential equation is both linear and separable so either method can be used to solve it. As you can check, the general solution is

$$u(t) = \sigma + Ce^{-kt}.$$

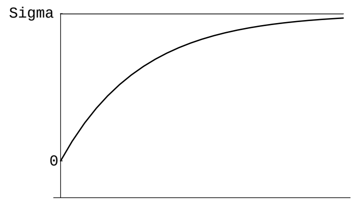
If the initial temperature of the object is $u(0) = u_0$, then

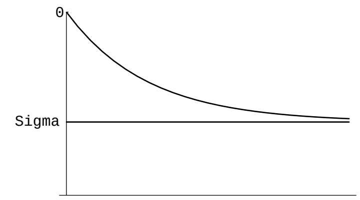
$$u_0 = \sigma + Ce^0 = \sigma + C \quad \text{and} \quad C = u_0 - \sigma.$$

Thus, the temperature of the object at any time t is given by

$$u(t) = \sigma + [u_0 - \sigma]e^{-kt}. \tag{2}$$

The graphs of $u(t)$ in the cases $u_0 < \sigma$ and $u_0 > \sigma$ are given below. Note that $\lim_{t \to \infty} u(t) = \sigma$ in each case. In the first case, u is increasing and its graph is concave down; in the second case, u is decreasing and its graph is concave up.





Example A metal bar with initial temperature $25^o\,C$ is dropped into a container of boiling water $(100^o\,C)$. After 5 seconds, the temperature of the bar is $35^o\,C$.

(a) What will the temperature of the bar be after 1 minute?

(b) How long will it take for the temperature of the bar to be within $0.5^o\,C$ of the boiling water?

SOLUTION Applying equation (2), the temperature of the bar at any time t is

$$T(t) = 100 + (25 - 100)\,e^{-kt} = 100 - 75\,e^{-kt}.$$

The first step is to determine the constant k. Since $T(5) = 35$, we have

$$35 = 100 - 75\,e^{-5k}, \quad 75\,e^{-5k} = 65, \quad -5k = \ln(65/75), \quad k \cong 0.0286.$$

Therefore,

$$T(t) = 100 - 75\,e^{-0.0286\,t}.$$

(a) The temperature of the bar after 1 minute is, approximately:

$$T(60) = 100 - 75\,e^{-0.0286(60)} \cong 100 - 75\,e^{-1.7172} \cong 86.53^o.$$

(b) We want to calculate how long it will take for the temperature of the bar to reach 99.5^o. Thus, we solve the equation

$$99.5 = 100 - 75\,e^{-0.0286\,t}$$

for t:

$$99.5 = 100 - 75\,e^{-0.0286\,t} \quad -75\,e^{-0.0286\,t} = -0.5, \quad -0.0826\,t = \ln(0.5/75), \quad t \cong 60.66\,\text{seconds}.$$ ■

Exercises 2.4.3

1. A thermometer is taken from a room where the temperature is $72^o\,F$ to the outside where the temperature is $32^o\,F$. After $1/2$ minute, the thermometer reads $50^o\,F$.

(a) What will the thermometer read after it has been outside for 1 minute?

(b) How many minutes does the thermometer have to be outside for it to read $35^o\,F$?

2. A metal ball at room temperature $20^o\,C$ is dropped into a container of boiling water ($100^o\,C$). given that the temperature of the ball increases 2^o in 2 seconds, find:

   (a) The temperature of the ball after 6 seconds in the boiling water.

   (b) How long it will take for the temperature of the ball to reach $90^o\,C$.

3. An object at a temperature of $50^o\,F$ is placed in an oven whose temperature is kept at $150^o\,F$. After 10 minutes, the temperature of the object is $75^o\,F$. Find:

   (a) An expression for the temperature of the object at any time t.

   (b) The time required for the object to reach $100^o\,F$.

   (c) The time required for the object to reach $200^o\,F$.

4. Suppose that a corpse is discovered at 10 p.m. and its temperature is determined to be $85^o\,F$. Two hours later, its temperature is $74^o\,F$. If the ambient temperature is $68^o\,F$, estimate the time of death.

5. An object with an initial temperature of $150^o\,C$ is placed in a room which is kept at a constant temperature of $35^o\,C$. The object's temperatures at 12:15 and 12:20 are $120^o\,C$ and $90^o\,C$, respectively.

   (a) At what time was the object placed in the room?

   (b) At what time will the object's temperature be $40^o\,C$?

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4. Falling Objects With Air Resistance

Consider an object with mass m in free fall near the surface of the earth. The object experiences the downward force of gravity (its weight) as well as air resistance, which may be modeled as a force that is proportional to velocity and acts in a direction opposite to the motion. By Newton's second law of motion, we have

$$m\,\frac{dv}{dt} = -m\,g - k\,v,$$

where g is the gravitational acceleration constant and $k > 0$ is the *drag coefficient* that depends upon the density of the atmosphere and aerodynamic properties of the object. (Note: If time is measured in seconds and distance in feet, then g is approximately 32 feet per second per second; if time is measured in seconds and distance in meters, then g is approximately 9.8 meters per second per second.) Rearrangement gives

$$\frac{dv}{dt} + r\,v = -g,$$

where $r = k/m$. Once this equation is solved for the velocity v, the height of the object is obtained by simple integration.

Exercises 2.4.4

1. (a) Solve the initial value problem
$$\frac{dv}{dt} + r\,v = -g, \quad v(0) = v_0$$
in terms of r, g, and v_0.
   (b) Show that $v(t) \to -mg/k$ as $t \to \infty$. This is called the *terminal velocity* of the object.
   (c) Integrate v to obtain the height y, assuming an initial height $y(0) = y_0$.

2. An object with mass 10 kg is dropped from a height of 200 m. Given that its drag coefficient is $k = 2.5$ N/(m/s), after how many seconds does the object hit the ground?

3. An object with mass 50 kg is dropped from a height of 200 m. It hits the ground 10 seconds later. Find the object's drag coefficient k.

4. An object with mass 10 kg is projected upward (from ground level) with initial velocity 60 m/s. It hits the ground 8.4 seconds later.
   (a) Find the object's drag coefficient k.
   (b) Find the maximum height.
   (c) Find the velocity with which the object hits the ground.

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5. Mixing Problems

Here we consider a tank, or other type of container, that contains a volume V of water in which some amount of impurity (e.g., salt) is dissolved. Water containing the dissolved impurity at a known concentration flows (or is pumped) into the tank at a given *volume flow rate*, and water flows out of the tank also at a given volume flow rate. We assume that the water in the tank remains thoroughly mixed at all times.

Let $A(t)$ be the amount of impurity in the tank at time t. The impurity concentration in the tank is then $A(t)/V$, and $A(t)$ will satisfy a differential equation of the generic form
$$\frac{dA}{dt} = (\text{inflow rate}) - (\text{outflow rate}).$$
These flow rates are products of the form
$$(\text{concentration}) \times (\text{volume flow rate}).$$

So if we let R_{in} and R_{out} denote the volume flow rates and let k_{in} denote the impurity concentration of the inflow, then we have

$$\frac{dA}{dt} = k_{\text{in}} R_{\text{in}} - \frac{A}{V}\, R_{\text{out}}.$$

When $R_{\text{in}} = R_{\text{out}}$, the volume V is constant. If $R_{\text{in}} \neq R_{\text{out}}$, but each is constant, then

$$V = V_0 + (R_{\text{in}} - R_{\text{out}})\, t.$$

Example A tank initially contains a solution of 10 pounds of pollutant in 60 gallons of water. Water with 1/2 pound of pollutant per gallon is added to the tank at the rate of 6 gal/min. The solution is thoroughly mixed and leaves the tank at the same rate. Find the amount $A(t)$ of pollutant in the tank at time t.

SOLUTION The inflow rate of pollutant is $k_{in} R_{in} = (1/2)6 = 3$ lbs/min and the outflow rate is

$$\frac{A}{60}\, R_{out} = \frac{A}{60} \cdot 6 = \frac{A}{10} \text{ lbs/min.}$$

Thus, the rate of change of pollutant in the tank at time t is

$$\frac{dA}{dt} = 3 - \frac{A}{10} \qquad \text{which can be written} \qquad \frac{dA}{dt} + \frac{1}{10} A = 3,$$

a linear equation. Also, we have $A(0) = 10$.

As you can check, the general solution of the differential equation is

$$A(t) = 30 + Ce^{-t/10}.$$

Applying the initial condition, we find that the amount of pollutant in the tank at time t is

$$A(t) = 30 - 20e^{-t/10}.$$

Exercises 2.4.5

1. A tank with a capacity of $2\,\text{m}^3$ (2000 liters) is initially full of pure water. At time $t = 0$, salt water with salt concentration 5 grams/liter begins to flow into the tank at a rate of 10 liters/minute. The well-mixed solution in the tank is pumped out at the same rate.

   (a) Set up, and then solve, the initial-value problem for the amount of salt in the tank at time t minutes.

   (b) Find the time when the salt concentration in the tank becomes 4 grams/liter.

2. A 100 gallon tank is initially full of water. At time $t = 0$, a 20% hydrochloric acid solution begins to flow into the tank at a rate of 2 gallons/minute. The well-mixed solution in the tank is pumped out at the same rate.

   (a) Set up, and then solve, the initial-value problem for the amount of hydrochloric acid in the tank at time t minutes.

   (b) Find the time when the hydrochloric acid concentration becomes 10%.

3. A room measuring $10\,\text{m} \times 5\,\text{m} \times 3\,\text{m}$ initially contains air that is free of carbon monoxide. At time $t = 0$, air containing 3% carbon monoxide enters the room at a rate of 1 m^3/minute, and the well-circulated air in the room leaves at the same rate.

   (a) Set up, and then solve, the initial-value problem for the amount of carbon monoxide in the room at time t minutes.

   (b) Find the time when the carbon monoxide concentration in the room reaches 2%.

4. A tank with a capacity of $1\,\text{m}^3$ (1000 liters) is initially half full of pure water. At time $t = 0$, 4% salt solution begins to flow into the tank at a rate of 30 liters/minute. The well-mixed solution in the tank is pumped out at a rate of 20 liters/minute.

   (a) Set up, and then solve, the initial-value problem for the amount of salt in the tank between time $t = 0$ and the time when the tank becomes full.

   (b) Find the salt *concentration* of the solution in the tank during this process.

5. A 100 gallon tank is initially full of pure water. At time $t = 0$, water containing salt at concentration 15 grams/gallon begins to flow into the tank at a rate of 1 gallon/minute, while the well-mixed solution in the tank is pumped out at a rate of 2 gallons/minute.

   (a) Set up, and then solve, the initial-value problem for the amount of salt in the tank between time $t = 0$ and the time when the tank becomes empty.

   (b) Find the maximum amount of salt in the tank during this process.

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6. The Logistic Equation

In the mid-nineteenth century the Belgian mathematician P.F. Verhulst used the differential equation

$$\frac{dy}{dt} = ky(M - y) \tag{1}$$

where k and M are positive constants, to study the population growth of various countries. This equation is now known as the *logistic equation* and its solutions are called *logistic functions*. Life scientists have used this equation to model the spread of an infectious disease through a population, and social scientists have used it to study the flow of information.

In the case of an infectious disease, if M denotes the number of people in the population and $y(t)$ is the number of infected people at time t, then the differential equation states that the rate of change of infected people is proportional to the product of the number of people who have the disease and the number of people who do not.

The constant M is called the *carrying capacity* of the environment. Note that $dy/dt > 0$ when $0 < y < M$, $dy/dt = 0$ when $y = M$, and $dy/dt < 0$ when $y > M$. The constant k is the *intrinsic growth rate*.

The differential equation (1) is separable and Bernoulli. We'll solve it as a separable equation. We write the equation as

$$\frac{1}{y(M-y)}\,y' = k$$

and integrate

$$\begin{aligned}
\int \frac{1}{y(M-y)}\,dy &= \int k\,dt + C_1 \\
\int \left(\frac{1/M}{y} + \frac{1/M}{M-y}\right) dy &= kt + C_1 \qquad \text{(partial fraction decomposition)} \\
\frac{1}{M}\ln|y| - \frac{1}{M}\ln|M-y| &= kt + C_1
\end{aligned}$$

We can solve this equation for y as follows:

$$\begin{aligned}
\frac{1}{M}\ln\left|\frac{y}{M-y}\right| &= kt + C_1 \\
\ln\left|\frac{y}{M-y}\right| &= Mkt + MC_1 = Mkt + C_2, \qquad (C_2 = MC_1) \\
\left|\frac{y}{M-y}\right| &= e^{Mkt+C_2} = e^{C_2}e^{Mkt} = Ce^{Mkt} \qquad (C = e^{C_2})
\end{aligned}$$

Now, in the context of this discussion, $y = y(t)$ satisfies $0 < y(t) < M$. Therefore, $y/(M-y) > 0$ and we have

$$\frac{y}{M-y} = C\,e^{Mkt}.$$

Solving this equation for y, we get

$$y(t) = \frac{CM}{C + e^{-Mkt}}.$$

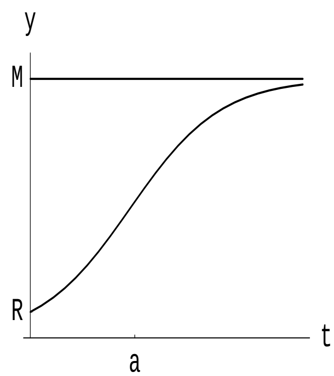
Finally, if $y(0) = R$, $R < M$, then

$$R = \frac{CM}{C+1} \qquad \text{which implies} \qquad C = \frac{R}{M-R}$$

and

$$y(t) = \frac{MR}{R + (M - R)e^{-Mkt}}. \tag{2}$$

The graph of this particular solution is shown below.



Note that y is an increasing function. In the Exercises you are asked to show that the graph is concave up on $[0, a)$ and concave down on $(a, \infty]$. This means that the disease is spreading at an increasing rate up to time a; after a, the disease is still spreading, but at a decreasing rate. Note, also, that $\lim_{t\to\infty} y(t) = M$.

Example An influenza virus is spreading through a small city with a population of 50,000 people. Assume that the virus spreads at a rate proportional to the product of the number of people who have been infected and the number of people who have not been infected. If 100 people were infected initially and 1000 were infected after 10 days, find:

(a) The number of people infected at any time t.

(b) How long it will take for half the population to be infected.

SOLUTION (a) Substituting the given data into equation (b), we have

$$y(t) = \frac{100(50,000)}{100 + 49,900\, e^{-50,000kt}} = \frac{50,000}{1 + 499\, e^{-50,000kt}}.$$

We can determine the constant k by applying the condition $y(10) = 1000$. We have

$$\begin{aligned} 1000 &= \frac{50,000}{1 + 499\, e^{-500,000k}} \\ 499\, e^{-500,000k} &= 49 \\ -500,000\, k &= \ln(49/499) \\ k &\cong 0.0000046. \end{aligned}$$

Thus, the number of people infected at time t is (approximately)

$$y(t) = \frac{50,000}{1 + 499\, e^{-0.23\, t}}.$$

(b) To find how long it will take for half the population to be infected, we solve $y(t) = 25,000$ for t:

$$\begin{aligned} 25,000 &= \frac{50,000}{1 + 499\, e^{-0.23\, t}} \\ 499\, e^{-0.23\, t} &= \frac{1}{499} \\ t &= \frac{\ln(1/499)}{-0.23} \cong 27 \text{ days.} \qquad \blacksquare \end{aligned}$$

Exercises 2.4.6

1. A rumor spreads through a small town with a population of 5,000 at a rate proportional to the product of the number of people who have heard the rumor and the number who have not heard it. Suppose that 100 people initiated the rumor and that 500 people heard it after 3 days.
   (a) How many people will have heard the rumor after 8 days?
   (b) How long will it take for half the population to hear the rumor.
2. A flu virus is spreading through a city with a population of 25,000. The disease is spreading at a rate proportional to the product of the number of people who have it and the number who don't. Suppose that 100 people had the flu initially and that 400 people had it after 10 days.
   (a) How many people will have the flu after 20 days?
   (b) How long will it take for half the population to have the flu?
3. Let y be the logistic function (4). Show that dy/dt increases for $y < M/2$ and decreases for $y > M/2$. What can you conclude about dy/dt when $y = M/2$?
4. Solve the logistic equation by means of the change of variables
   $$y(t) = v(t)^{-1}, \quad y'(t) = -v(t)^{-2}\, v'(t).$$
   Express the constant of integration in terms of the initial value $y(0) = y_0$.
5. Suppose that a population governed by a logistic model exists in an environment with carrying capacity of 800. If an initial population of 100 grows to 300 in 3 years, find the intrinsic growth rate k.