2.2 Separable Differential Equations

SEPARABLE DIFFERENTIAL EQUATION A first order differential equation $y' = f(x, y)$ is a *separable equation* if the function f can be expressed as the product of a function of x and a function of y. That is, the equation is separable if the function f has the form

$$f(x, y) = p(x)\, h(y).$$

where p and h are continuous functions.

The solution method for separable equations is based on writing the equation as

$$\frac{1}{h(y)}\, y' = p(x)$$

or

$$q(y)\, y' = p(x) \tag{1}$$

where $q(y) = 1/h(y)$.

Of course, in dividing the equation by $h(y)$ we have to assume that $h(y) \neq 0$. Any numbers r such that $h(r) = 0$ **may** result in *singular solutions* of the form $y(x) \equiv r$.

If we write y' as dy/dx and interpret this symbol as "differential y" divided by "differential x," then a separable equation can be written in differential form as

$$q(y)\, dy = p(x)\, dx.$$

This is the motivation for the term "separable," the variables are separated.

Solution Method for Separable Equations

Step 1. Identify: Can you write the equation in the form (1). If yes, do so.

In expanded form, equation (1) is

$$q(y(x))\, y'(x) = p(x).$$

Step 2. Integrate this equation with respect to x:

$$\int q(y(x))\, y'(x)\, dx = \int p(x)\, dx + C \qquad C \text{ an arbitrary constant}$$

which can be written

$$\int q(y)\, dy = \int p(x)\, dx + C$$

by setting $y = y(x)$ and $dy = y'(x)\, dx$. Now, if P is an antiderivative for p, and if Q is an antiderivative for q, then this equation is equivalent to

$$Q(y) = P(x) + C. \tag{2}$$

INTEGRAL CURVES Equation (2) is a one-parameter family of curves called the *integral curves* of equation (1). In general, the integral curves define y implicitly as a function of x. These curves are solutions of (1) since, by implicit differentiation,

$$\begin{aligned} \frac{d}{dx}\left[Q(y)\right] &= \frac{d}{dx}[P(x)] + \frac{d}{dx}[C] \\ \text{and} \quad q(y(x))\, y'(x) &= p(x) \quad \text{or} \quad q(y)y' = p(x). \end{aligned}$$

Example 1. The differential equation

$$y' = -\frac{x}{y} \qquad (y \neq 0)$$

is separable since $f(x,y) = -(x/y) = (-x)(1/y)$. Writing the equation in the form (1)

$$y\, y' = -x \qquad \text{or} \qquad y\, dy = -x\, dx$$

and integrating

$$\int y\, dy = -\int x\, dx + C,$$

we get

$$\tfrac{1}{2}\, y^2 = -\tfrac{1}{2}\, x^2 + C \qquad \text{or} \qquad \tfrac{1}{2}\, x^2 + \tfrac{1}{2}\, y^2 = C$$

which, after multiplying by 2, gives

$$x^2 + y^2 = 2C \quad \text{or} \quad x^2 + y^2 = C.$$

(Since C is an arbitrary constant, $2C$ is arbitrary and so we'll just call it C again. This "manipulation" of arbitrary constants is standard in differential equations courses; you'll have to become accustomed to it.)

The set of integral curves is the family of circles centered at the origin. Note that for each positive value of C, the resulting equation defines y implicitly as a function of x. ■

Remark we may or may not be able to solve the implicit relation (2) for y. This is in contrast to linear differential equations where the solutions $y = y(x)$ are given explicitly as a function of x. (See equation (2) in Section 2.1.) When we can solve (2) for y, we will. ■

As we shall illustrate below, the set of integral curves of a separable equation *may not* represent the set of all solutions of the equation and so it is not technically correct to use the term "general solution" as we did with linear equations. However for our purposes here this is a minor point and so we shall also call (2) the general solution of (1). As noted above, if $h(r) = 0$, then $y(x) \equiv r$ may be a singular solution of the equation; we will have to check for singular solutions when solving separable equations.

Example 2. Show that the differential equation

$$y' = \frac{xy - y}{y + 1} \qquad (y \neq -1)$$

is separable. Then

1. Find the general solution and any singular solutions.
2. Find a solution which satisfies the initial condition $y(2) = 1$.

SOLUTION Here

$$f(x, y) = \frac{xy - y}{y + 1} = \frac{y(x-1)}{y+1} = (x-1)\,\frac{y}{y+1}.$$

Thus, f can be expressed as the product of a function of x and a function of y so the equation is separable.

Writing the equation in the form (1), we have

$$\frac{y+1}{y}\,y' = x - 1 \qquad (y \neq 0)$$

or

$$\left(1 + \frac{1}{y}\right) y' = x - 1 \qquad \text{and} \qquad \left(1 + \frac{1}{y}\right) dy = (x-1)\,dx;$$

the variables are separated. Integrating with respect to x, we get

$$\int \left(1 + \frac{1}{y}\right) dy = \int (x-1)\,dx + C$$

and

$$y + \ln|y| = \tfrac{1}{2}\,x^2 - x + C$$

is the general solution. Again we have y defined implicitly as a function of x.

Note that $y(x) \equiv 0$ is a solution of the differential equation (verify this), but this function is not included in the general solution (ln 0 does not exist). Thus, $y(x) \equiv 0$ is a singular solution of the equation.

To find a solution that satisfies the initial condition, set $x = 2,\ y = 1$ in the general solution:

$$1 + \ln 1 = \tfrac{1}{2}\,(2)^2 - 2 + C \quad \text{which implies} \quad C = 1.$$

A particular solution that satisfies the initial condition is: $y + \ln|y| = \frac{1}{2}\,x^2 - x + 1.$ ■

Example 3. Show that the differential equation

$$y' = xe^{y-x}$$

is separable, and find the general solution and any singular solutions.

SOLUTION Since $e^{y-x} = e^{-x}e^{y}$ we have

$$f(x,y) = xe^{y-x} = xe^{-x}e^{y}.$$

Thus, f can be expressed as the product of a function of x and a function of y; the equation is separable.

We have

$$y' = xe^{y-x} = xe^{-x}e^{y}.$$

To write the equation in the form (1), divide by e^{y} (i.e., multiply by e^{-y}) to get

$$e^{-y}y' = xe^{-x} \qquad \text{and} \qquad e^{-y}\,dy = xe^{-x}\,dx.$$

(**NOTE:** Since $e^{-y} \neq 0$ for all y, there will be no singular solutions.)

Integrating with respect to x we get

$$\begin{aligned} \int e^{-y}\,dy &= \int xe^{-x}\,dx + C \\ -e^{-y} &= -xe^{-x} - e^{-x} + C \quad \text{(integration by parts)} \\ e^{-y} &= xe^{-x} + e^{-x} + C \quad \text{(the general solution).} \end{aligned}$$

Again, the general solution gives y implicitly as a function of x. However, in this case we can solve for y by applying the natural log function:

$$\begin{aligned} -y &= \ln\left(xe^{-x} + e^{-x} + C\right) \quad \text{and} \\ y(x) &= -\ln\left(xe^{-x} + e^{-x} + C\right) = \ln\left(\frac{1}{xe^{-x} + e^{-x} + C}\right). \qquad \blacksquare \end{aligned}$$

Example 4. As you can check, the differential equation

$$y' = xy + 2x$$

is both linear and separable so it can be solved using either method. We'll solve it as a separable equation. You should also solve it as a linear equation and compare the two approaches.

Rewriting the equation in the form (1), we have

$$\frac{1}{y+2}\,y' = x \qquad (y \neq -2).$$

Integrating this equation, we get

$$\begin{aligned}
\int \frac{1}{y+2}\,dy &= \int x\,dx + C \\
\ln|\,y+2| &= \tfrac{1}{2}\,x^2 + C \quad \text{(general solution)}
\end{aligned}$$

We can solve the latter equation for y as follows:

$$\begin{aligned}
\ln|\,y+2| &= \tfrac{1}{2}\,x^2 + C \\
|\,y+2| &= e^{x^2/2+C} = e^C\,e^{x^2/2} = K\,e^{x^2/2} \quad (K = e^C,\ \text{(an arbitrary constant)}
\end{aligned}$$

By allowing K to take on both positive and negative values, we can write the general solution as

$$y + 2 = K\,e^{x^2/2} \quad \text{or} \quad y = K\,e^{x^2/2} - 2.$$

If we set $K = 0$ we get $y = -2$ so this solution is included in the general solution; $y(x) \equiv -2$ is not a singular solution. ■